

Characterizations of Super-regularity and its Variants

Aris Daniilidis* D. Russell Luke† Matthew K. Tam‡

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Abstract

Convergence of projection-based methods for nonconvex set feasibility problems has been established for sets with ever weaker regularity assumptions. What has not kept pace with these developments is analogous results for convergence of optimization problems with correspondingly weak assumptions on the value functions. Indeed, one of the earliest classes of nonconvex sets for which convergence results were obtainable, the class of so-called *super-regular sets* [10], has no functional counterpart. In this work, we amend this gap in the theory by establishing the equivalence between a property slightly stronger than super-regularity, which we call *Clarke super-regularity*, and *subsmoothness* of sets as introduced by Aussel, Daniilidis and Thibault [1]. The bridge to functions shows that approximately convex functions studied by Ngai, Luc and Thera [12] are those which have Clarke super-regular epigraphs. Further classes of regularity of functions based on the corresponding regularity of their epigraph are also discussed.

Keywords. super-regularity, subsmoothness, approximately convex

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1 Introduction

The notion of a *super-regular* set was introduced by Lewis, Luke and Malick [10] who recognized the property as an important ingredient for proving convergence of the method of alternating projections without convexity. This was generalized in subsequent publications [3, 6, 7, 11], with the weakest known assumptions guaranteeing local linear convergence of the alternating projections algorithm for two-set, consistent feasibility problems to date found in [15, Theorem 3.3.5]. The regularity assumptions on the individual sets in these subsequent works are vastly weaker than super-regularity, but what has not kept pace with these generalizations is their functional analogs. Indeed, it appears that

*DIM-CMM, Universidad de Chile, 8370459 Chile. E-Mail: arisd@dim.uchile.cl

†Institut für Numerische und Angewandte Mathematik, Universität Göttingen, 37083 Göttingen, Germany. E-Mail: r.luke@math.uni-goettingen.de

‡Institut für Numerische und Angewandte Mathematik, Universität Göttingen, 37083 Göttingen, Germany. E-Mail: m.tam@math.uni-goettingen.de

the notion of a *super-regular function* has not yet been articulated. In this note, we bridge this gap between super-regularity of sets and functions as well as establishing connections to other known function-regularities in the literature. A missing link is yet another type of set regularity, what we call *Clarke super-regularity*, which is a slightly stronger version of super-regularity and, as we show, this is equivalent to other existing notions of regularity. For a general set that is not necessarily the epigraph of a function, we establish an equivalence between *subsmoothness* as introduced by Aussel, Daniilidis and Thibault [1] and Clarke super-regularity.

To begin, in Section 2 we recall different concepts of the normal cones to a set as well as notions of set regularity, including Clarke regularity (Definition 2.3) and (limiting) super-regularity (Definition 2.4). Next, in Section 3 we introduce the notion of Clarke super-regularity (Definition 3.1) and relate it to the notion of subsmoothness (Theorem 3.4). We also provide an example illustrating that Clarke super-regularity at a point is a strictly weaker condition than Clarke regularity around the point (Example 3.2). Finally, in Section 4, we provide analogous

statements for Lipschitz continuous functions, relating the class of approximately convex functions to super-regularity of the epigraph. After completing this work we received a preprint [16] which contains results of this flavor, including a characterization of (limiting) super-regularity in terms of (metric) subsmoothness.

2 Normal cones and Clarke regularity

The notation used throughout this work is standard for the field of variational analysis, as can be found in [14]. The *closed ball* of radius $r > 0$ centered at $x \in \mathbb{R}^n$ is denoted $\mathbb{B}_r(x)$ and the closed unit ball is denoted $\mathbb{B} := \mathbb{B}_1(0)$. The *(metric) projector* onto a set $\Omega \subset \mathbb{R}^n$, denoted by $P_\Omega : \mathbb{R}^n \rightrightarrows \Omega$, is the multi-valued mapping defined by

$$P_\Omega(x) := \{\omega \in \Omega : \|x - \omega\| = d(x, \Omega)\},$$

where $d(x, \Omega)$ denotes the distance of the point $x \in \mathbb{R}^n$ to the set Ω . When Ω is nonempty and closed, its projector P_Ω is everywhere nonempty. A selection from the projector is called a *projection*.

Given a set Ω , we denote its *closure* by $\text{cl}\Omega$, its *convex hull* by $\text{conv}\Omega$, and its *conic hull* by $\text{cone}\Omega$. In this work we shall deal with two fundamental tools in nonsmooth analysis; *normal cones* to sets and *subdifferentials* of functions (Section 4).

Definition 2.1 (normal cones). *Let $\Omega \subseteq \mathbb{R}^n$ and let $\bar{\omega} \in \Omega$.*

(i) *The proximal normal cone of Ω at $\bar{\omega} \in \Omega$ is defined by*

$$N_\Omega^P(\bar{\omega}) = \text{cone}(P_\Omega^{-1}\bar{\omega} - \bar{\omega}).$$

Equivalently, $(\bar{\omega}, \bar{\omega}^) \in \text{gph}N_\Omega^P$, or simply, $\bar{\omega}^* \in N_\Omega^P(\bar{\omega})$, whenever there exists $\sigma \geq 0$ such that*

$$\langle \bar{\omega}^*, \omega - \bar{\omega} \rangle \leq \sigma \|\omega - \bar{\omega}\|^2, \quad \forall \omega \in \Omega.$$

The A -restricted proximal normal cone of ω at $\bar{\omega}$ is

$$N_{\Omega|A}^P(\bar{\omega}) := \text{cone} \left((A \cap P_{\Omega}^{-1}\bar{\omega}) - \bar{\omega} \right).$$

(ii) The Fréchet normal cone of Ω at $\bar{\omega}$ is defined by

$$\hat{N}_{\Omega}(\bar{\omega}) = \{ \bar{\omega}^* \in \mathbb{R}^n : \langle \bar{\omega}^*, \omega - \bar{\omega} \rangle \leq o(\|\omega - \bar{\omega}\|), \forall \omega \in \Omega \},$$

Equivalently, $\bar{\omega}^* \in \hat{N}_{\Omega}(\bar{\omega})$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\langle \bar{\omega}^*, \omega - \bar{\omega} \rangle \leq \varepsilon \|\omega - \bar{\omega}\|, \quad \text{for all } \omega \in \Omega \cap \mathbb{B}_{\delta}(\bar{\omega}). \quad (1)$$

(iii) The limiting normal cone of Ω at $\bar{\omega}$ is defined by

$$N_{\Omega}(\bar{\omega}) = \text{Lim sup}_{\omega \rightarrow \bar{\omega}} \hat{N}_{\Omega}(\omega),$$

where the limit superior denotes the Painlevé–Kuratowski outer limit.

(iv) The Clarke normal cone of Ω at $\bar{\omega}$ is defined by

$$N_{\Omega}^C(\bar{\omega}) = \text{cl conv } N_{\Omega}(\bar{\omega}).$$

When $\bar{\omega} \notin \Omega$, all of the aforementioned normal cones at $\bar{\omega}$ are defined to be empty.

The restricted normal cone was first introduced in [3, Definition 2.1] and is used in Example 2.7.

Central to our subsequent analysis is the notion of a *truncation* of a normal cone. Given $r > 0$, one defines the r -truncated version of each of the above cones to be its intersection with a ball centered at the origin of radius r . For instance, the r -truncated proximal normal cone of Ω at $\bar{\omega} \in \Omega$ is defined by

$$N_{\Omega}^{\text{rP}}(\bar{\omega}) = \text{cone} (P_{\Omega}^{-1}\bar{\omega} - \bar{\omega}) \cap \mathbb{B}_r,$$

that is, $\bar{\omega}^* \in N_{\Omega}^{\text{rP}}(\bar{\omega})$ whenever $\|\bar{\omega}^*\| \leq r$ and for some $\sigma \geq 0$ we have

$$\langle \bar{\omega}^*, \omega - \bar{\omega} \rangle \leq \sigma \|\omega - \bar{\omega}\|^2, \quad \forall \omega \in \Omega.$$

In general, the following inclusions between the normal cones can deduce straightforwardly from their respective definitions:

$$N_{\Omega}^P(\bar{\omega}) \subseteq \hat{N}_{\Omega}(\bar{\omega}) \subseteq N_{\Omega}(\bar{\omega}) \subseteq N_{\Omega}^C(\bar{\omega}). \quad (2)$$

The regularity of sets is characterized by the relation between elements in the graph of the normal cones to the sets and directions constructable from points in the sets. The weakest kind of regularity of sets that has been shown to guarantee convergence of the alternating projections algorithm is *elemental subregularity* (see [7, Cor.3.13(a)] and [15, Theorem 3.3.5]). It was called *elemental* (sub)regularity in [8, Definition 5]

and [11, Definition 3.1] to distinguish regularity of sets from regularity of collections of sets. Since we are only considering the regularity of sets, and later functions, we can drop the “elemental” qualifier in the present setting. We also streamline the terminology and variations on elemental subregularity used in [8, 11], replacing *uniform* elemental subregularity with a more versatile and easily distinguishable variant.

Definition 2.2 (subregularity [8, Definition 5]). *Let $\Omega \subseteq \mathbb{R}^n$ and $\bar{\omega} \in \Omega$. The set Ω is said to be ε -subregular relative to Λ at $\bar{\omega}$ for $(\hat{\omega}, \hat{\omega}^*) \in \text{gph } N_\Omega$ if it is locally closed at $\bar{\omega}$ and there exists an $\varepsilon > 0$ together with a neighborhood U of $\bar{\omega}$ such that*

$$\langle \hat{\omega}^* - (\omega' - \omega), \omega - \hat{\omega} \rangle \leq \varepsilon \|\hat{\omega}^* - (\omega' - \omega)\| \|\omega - \hat{\omega}\|, \quad \forall \omega' \in \Lambda \cap U, \quad \forall \omega \in P_\Omega(\omega'). \quad (3)$$

If for every $\varepsilon > 0$ there is a neighborhood (depending on ε) such that (3) holds, then Ω is said to be subregular relative to Λ at $\bar{\omega}$ for $(\hat{\omega}, \hat{\omega}^) \in \text{gph } N_\Omega$.*

The property that distinguishes the degree of regularity of sets is the diversity of vectors $(\hat{\omega}, \hat{\omega}^*) \in \text{gph } N_\Omega$ for which (3) holds, as well as the choice of the set Λ . Of particular interest to us are *Clarke regular* sets, which satisfy (3) for all $\varepsilon > 0$ and for all Clarke normal vectors at $\bar{\omega}$.

Definition 2.3 (Clarke regularity). *The set Ω is said to be Clarke regular at $\bar{\omega} \in \Omega$ if it is locally closed at $\bar{\omega}$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $(\bar{\omega}, \bar{\omega}^*) \in \text{gph } N_\Omega^C$*

$$\langle \bar{\omega}^*, \omega - \bar{\omega} \rangle \leq \varepsilon \|\bar{\omega}^*\| \|\omega - \bar{\omega}\|, \quad \forall \omega \in \Omega \cap \mathbb{B}_\delta(\bar{\omega}). \quad (4)$$

Note that (4) is (3) with $\Lambda = \Omega$ and $U = \mathbb{B}_\delta(\bar{\omega})$, which in the case of Clarke regularity holds for all $(\bar{\omega}, \bar{\omega}^*) \in \text{gph } N_\Omega^C$. A short argument shows that, for Ω Clarke regular at $\bar{\omega}$, the Clarke and Fréchet normal cones coincide at $\bar{\omega}$. Indeed, this property is used to *define* Clarke regularity in [14, Definition 6.4]. It is also immediately clear from the definitions that if Ω is Clarke regular at $\bar{\omega}$, then it is subregular relative to $\Lambda = \Omega$ at $\bar{\omega}$ for all $\bar{\omega}^* \in N_\Omega(\bar{\omega})$.

By setting $\Lambda = \mathbb{R}^n$, letting $\hat{\omega} \in \Omega$ be in a neighborhood of $\bar{\omega}$ and fixing $\hat{\omega}^* = 0$ in the context of Definition 2.2, we arrive at super-regularity which, when stated explicitly, takes the following form.

Definition 2.4 (super-regularity [10, Definition 4.3]). *Let $\Omega \subseteq \mathbb{R}^n$ and $\bar{\omega} \in \Omega$. The set Ω is said to be super-regular at $\bar{\omega}$ if it is locally closed at $\bar{\omega}$ and for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $(\hat{\omega}, 0) \in \text{gph } N_\Omega \cap \{(\mathbb{B}_\delta(\bar{\omega}), 0)\}$*

$$\langle \omega' - \omega, \hat{\omega} - \omega \rangle \leq \varepsilon \|\omega' - \omega\| \|\hat{\omega} - \omega\|, \quad \forall \omega' \in \mathbb{B}_\delta(\bar{\omega}), \quad \forall \omega \in P_\Omega(\omega'). \quad (5)$$

Rewriting the above leads to the following equivalent characterization of super-regularity, which is more useful for our purposes.

Proposition 2.5 ([10, Proposition 4.4]). *The set $\Omega \subseteq \mathbb{R}^n$ is super-regular at $\bar{\omega} \in \Omega$ if and only if it is locally closed at $\bar{\omega}$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\begin{aligned} \langle \omega_1^*, \omega_2 - \omega_1 \rangle &\leq \varepsilon \|\omega_1^*\| \|\omega_2 - \omega_1\|, \\ \forall (\omega_1, \omega_1^*) &\in \text{gph } N_\Omega \cap (\mathbb{B}_\delta(\bar{\omega}) \times \mathbb{R}^n), \quad \forall \omega_2 \in \Omega \cap \mathbb{B}_\delta(\bar{\omega}). \end{aligned} \quad (6)$$

It is immediately clear from this characterization that super-regularity implies Clarke regularity. By continuing our development of increasingly nicer regularity properties to convexity, we have the following relationships involving stronger notions of regularity.

Proposition 2.6. *Let $\Omega \subseteq \mathbb{R}^n$ be locally closed at $\bar{\omega} \in \Omega$.*

(i) *If Ω is prox-regular at $\bar{\omega}$ (i.e., there exists a neighborhood of $\bar{\omega}$ on which the projector is single-valued), then there is a constant $\gamma > 0$ such that for all $\varepsilon > 0$*

$$\begin{aligned} \langle \omega_1^*, \omega_2 - \omega_1 \rangle &\leq \varepsilon \|\omega_1^*\| \|\omega_2 - \omega_1\|, \\ \forall (\omega_1, \omega_1^*) &\in \text{gph } N_\Omega \cap (\mathbb{B}_{\gamma\varepsilon}(\bar{\omega}) \times \mathbb{R}^n), \quad \forall \omega_2 \in \Omega \cap \mathbb{B}_{\gamma\varepsilon}(\bar{\omega}). \end{aligned} \quad (7)$$

(ii) *If Ω is convex, then*

$$\langle \omega_1^*, \omega_2 - \omega_1 \rangle \leq 0, \quad \forall (\omega_1, \omega_1^*) \in \text{gph } N_\Omega, \quad \forall \omega_2 \in \Omega. \quad (8)$$

Proof. The proof of (i) can be found in [8, Proposition 4(vi)]. Part (ii) is classical. \square

Example 2.7 (Pac-Man). Let $\bar{x} = 0 \in \mathbb{R}^2$ and consider two subsets of \mathbb{R}^2 given by

$$\begin{aligned} A &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1, -(1/2)x_1 \leq x_2 \leq x_1, x_1 \geq 0\}, \\ B &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1, x_1 \leq |x_2|\}. \end{aligned}$$

The set B looks like a ‘‘Pac-Man’’ with mouth opened to the right and the set A , if you like, a piece of pizza. For an illustration, see Figure 1.

The set B is subregular relative to A at $\bar{x} = 0$ for all $(b, v) \in \text{gph} \left(N_{B|A}^{\text{P}} \right)$ (the A -restricted proximal normal cone, see Definition 2.1) with $\varepsilon = 0$ on all neighborhoods. Indeed, for all $a \in A$, $a_B \in P_B(a)$ and $v \in N_{B|A}^{\text{P}}(b)$, we have

$$\langle v - (a - a_B), a_B - b \rangle = \langle v, a_B - b \rangle - \langle a - a_B, a_B - b \rangle = 0.$$

In other words, from the perspective of the piece of pizza, Pac-Man looks *convex*.

The set B , however, is not ε -subregular for any $\varepsilon < 1$ at $\bar{x} = 0$ relative to \mathbb{R}^2 for any $(0, v) \in \text{gph } N_B$. Indeed, by choosing $x = tv \in B$ (where $0 \neq v \in B \cap N_B(0)$, $t \downarrow 0$), we have $\langle v, x \rangle = \|v\| \|x\| > 0$. Clearly, this also means that Pac-Man is not Clarke regular. \diamond

3 Super-regularity and subsmoothness

In the context of the definitions surveyed in the previous section, we introduce an even stronger type of regularity that we identify, in Theorem 3.4, with *subsmoothness* as studied in [1]. This will provide a crucial link to the analogous characterizations of regularity for *functions* considered in Theorem 4.6, in particular, to *approximately convex* functions studied in [12].

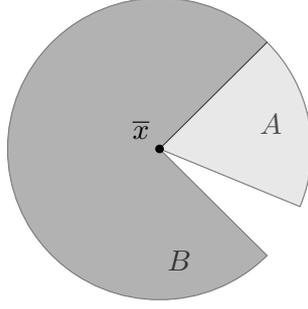


Figure 1: An illustration of the sets in Example 2.7.

Definition 3.1 (Clarke super-regularity). *Let $\Omega \subseteq \mathbb{R}^n$ and $\bar{\omega} \in \Omega$. The set Ω is said to be Clarke super-regular at $\bar{\omega}$ if it is locally closed at $\bar{\omega}$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $(\hat{\omega}, \hat{\omega}^*) \in \text{gph } N_{\Omega}^{\text{C}} \cap (\mathbb{B}_{\delta}(\bar{\omega}) \times \mathbb{R}^n)$, the following inequality holds*

$$\langle \hat{\omega}^*, \omega - \hat{\omega} \rangle \leq \varepsilon \|\hat{\omega}^*\| \|\omega - \hat{\omega}\|, \quad \forall \omega \in \Omega \cap \mathbb{B}_{\delta}(\bar{\omega}). \quad (9)$$

The only difference between Clarke super-regularity and super-regularity is that, in the case of Clarke super-regularity, the key inequality above holds for all nonzero Clarke normals in a neighborhood instead holding only for limiting normals (compare (6) with (9)). It therefore follows that Clarke super-regularity at a point implies Clarke regularity there. Nevertheless, even this stronger notion of regularity does not imply Clarke regularity around $\bar{\omega}$, as the following counterexample shows.

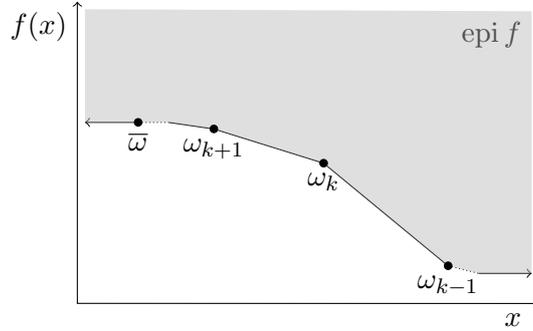


Figure 2: A sketch of the function f and the sequence (ω_k) given in Example 3.2.

Example 3.2 (regularity only at a point). Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the continuous, piecewise linear function (see Figure 2) defined by

$$f(x) := \begin{cases} 0, & \text{if } x \leq 0 \\ -\frac{1}{2^{k+1}}(x - \frac{1}{2^k}) - \frac{1}{3 \cdot 4^k}, & \text{if } \frac{1}{2^{k+1}} \leq x \leq \frac{1}{2^k} \quad (\text{for } k = 1, 2, \dots) \\ -\frac{1}{12}, & \text{if } x \geq \frac{1}{2}. \end{cases}$$

Notice that

$$-\frac{4}{3}x^2 \leq f(x) \leq -\frac{1}{3}x^2, \quad \forall x \in \left[0, \frac{1}{2}\right]. \quad (10)$$

Let $\Omega = \text{epi } f$ denote the epigraph of f . Thanks to (10) it is easily seen that Ω is Clarke regular at $\bar{\omega} = (0, 0)$ in the sense of Definition 2.3. However, Ω is not Clarke regular at the sequence of points $\omega_k = (\frac{1}{2^{k+1}}, \frac{1}{2^k})$ converging to $\bar{\omega}$. Indeed, the Fréchet normal cones $\hat{N}_\Omega(\omega_k)$ are reduced to $\{0\}$ for all $k \geq 1$, while the corresponding limiting normal cones are given by

$$N_\Omega(\omega_k) = \mathbb{R}_+ \left\{ \left(-\frac{1}{2^k}, -1 \right), \left(-\frac{1}{2^{k+1}}, -1 \right) \right\}, \quad \forall k \in \mathbb{N}.$$

◇

A missing link in the cascade of set regularity is subsmooth and semi-subsmooth sets, introduced and studied by Aussel, Daniilidis and Thibault in [1, Definitions 3.1 & 3.2].

Definition 3.3 ((Semi-)subsmooth sets). *Let $\Omega \subset \mathbb{R}^n$ be closed and let $\bar{\omega} \in \Omega$.*

- (i) *The set Ω is subsmooth at $\bar{\omega} \in \Omega$ if, for every $r > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\omega_1, \omega_2 \in \mathbb{B}_\delta(\bar{\omega}) \cap \Omega$, all $\omega_1^* \in N_\Omega^{rC}(\omega_1)$ and all $\omega_2^* \in N_\Omega^{rC}(\omega_2)$ we have:*

$$\langle \omega_1^* - \omega_2^*, \omega_1 - \omega_2 \rangle \geq -\varepsilon \|\omega_1 - \omega_2\|. \quad (11)$$

- (ii) *The set Ω is semi-subsmooth at $\bar{\omega}$ if, for every $r > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\omega \in \mathbb{B}_\delta(\bar{\omega}) \cap \Omega$, all $\omega^* \in N_\Omega^{rC}(\omega)$ and all $\bar{\omega}^* \in N_\Omega^{rC}(\bar{\omega})$*

$$\langle \omega^* - \bar{\omega}^*, \omega - \bar{\omega} \rangle \geq -\varepsilon \|\omega - \bar{\omega}\|. \quad (12)$$

It is clear from the definitions that subsmoothness at a point implies semi-subsmoothness at the same point. The next theorem establishes the precise connection between subsmoothness and Clarke super-regularity (Definition 3.1).

Theorem 3.4 (characterization of subsmoothness). *Let $\Omega \subseteq \mathbb{R}^n$ be closed and nonempty.*

- (i) *The set Ω is subsmooth at $\bar{\omega} \in \Omega$ if and only if Ω is Clarke super-regular at $\bar{\omega}$.*
- (ii) *The set Ω is semi-subsmooth at $\bar{\omega} \in \Omega$ if and only if for each constant $\varepsilon > 0$ there is a $\delta > 0$ such that for every $(\bar{\omega}, \bar{\omega}^*) \in \text{gph } N_\Omega^C$*

$$\langle \bar{\omega}^*, \omega - \bar{\omega} \rangle \leq \varepsilon \|\bar{\omega}^*\| \|\omega - \bar{\omega}\|, \quad \forall \omega \in \Omega \cap \mathbb{B}_\delta(\bar{\omega})$$

and for all $(\omega, \omega^) \in \text{gph } N_\Omega^C \cap (\mathbb{B}_\delta(\bar{\omega}) \times \mathbb{R}^n)$,*

$$\langle \omega^*, \bar{\omega} - \omega \rangle \leq \varepsilon \|\omega^*\| \|\bar{\omega} - \omega\|.$$

Proof. (i). Assume Ω is subsmooth at $\bar{\omega} \in \Omega$ and fix an $\varepsilon > 0$. Set $r = 1$ and let $\delta > 0$ be given by the definition of subsmoothness. Then for every $\omega_1, \omega_2 \in \Omega \cap \mathbb{B}_\delta(\bar{\omega})$ and $\omega_2^* \in N_\Omega^C(\omega_2) \setminus \{0\}$, applying (11) for $\omega_1^* = 0 \in N_\Omega^{(r=1)C}(\omega_1)$ and $\|\omega_2^*\|^{-1}\omega_2^* \in N_\Omega^{(r=1)C}(\omega_2)$ we deduce (9). The same argument applies in the case that $\omega_2^* = 0$ and $\omega_1^* \neq 0$. If both $\omega_1^* = \omega_2^* = 0$, then the required inequality holds trivially.

Let us now assume that Ω is Clarke super-regular at $\bar{\omega}$ and fix $r > 0$ and $\varepsilon > 0$. Let $\delta > 0$ be given by the definition of Clarke super-regularity corresponding to $\varepsilon' = \varepsilon/2r$ and let $\omega_1, \omega_2 \in \mathbb{B}_\delta(\bar{\omega}) \cap \Omega$, $\omega_1^* \in N_\Omega^{rC}(\omega_1)$ and $\omega_2^* \in N_\Omega^{rC}(\omega_2)$. It follows from (9) that

$$\begin{aligned} \langle \omega_1^*, \omega_1 - \omega_2 \rangle &\geq -\frac{\varepsilon}{2r} \|\omega_1^*\| \|\omega_1 - \omega_2\| \geq -\frac{\varepsilon}{2} \|\omega_1 - \omega_2\| \\ &\text{and} \\ \langle -\omega_2^*, \omega_1 - \omega_2 \rangle &\geq -\frac{\varepsilon}{2r} \|\omega_2^*\| \|\omega_1 - \omega_2\| \geq -\frac{\varepsilon}{2} \|\omega_1 - \omega_2\|. \end{aligned}$$

We conclude by adding the above inequalities.

Part (ii) is nearly identical and the proof is omitted. \square

The following corollary utilizes Theorem 3.4 to summarize the relations between various notions of regularity for sets, the weakest of these being the weakest known regularity assumption under which local convergence of alternating projections has been established [15, Theorem 3.3.5].

Corollary 3.5. *Let $\Omega \subseteq \mathbb{R}^n$ be closed, let $\bar{\omega} \in \Omega$ and consider the following assertions.*

- (i) Ω is prox-regular at $\bar{\omega}$.
- (ii) Ω is subsmooth at $\bar{\omega}$.
- (iii) Ω is Clarke super-regular at ω .
- (iv) Ω is (limiting) super-regular at ω .
- (v) Ω is Clarke regular at ω .
- (vi) Ω is subregular at ω relative to some nonempty $\Lambda \subset \mathbb{R}^n$ for all $(\omega, \omega^*) \in V \subset \text{gph } N_\Omega^P$.

Then (i) \implies (ii) \iff (iii) \implies (iv) \implies (v) \implies (vi).

Proof. (i) \implies (ii): This was shown in [1, Proposition 3.4(ii)]. (ii) \iff (iii): This is Theorem 3.4(i). (iii) \implies (iv) \implies (v) \implies (vi): These implications follow from the definitions. \square

Remark 3.6 (amenability). A further regularity between convexity and prox-regularity is *amenability* [14, Definition 10.23]. This was shown in [13, Corollary 2.12] to imply prox-regularity. Amenability plays a larger role in the analysis of functions and is defined precisely in this context below. \diamond

4 Regularity of functions

The extension of the above notions of set regularity to analogous notions for functions typically passes through the epigraphs. Given a function $f : \mathbb{R}^n \rightarrow [-\infty, +\infty]$, recall that its *domain* is $\text{dom } f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$ and its *epigraph* is

$$\text{epi } f := \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq \alpha\}.$$

The *subdifferential* of a function at a point \bar{x} can be defined in terms of the normal cone to its epigraph at $(\bar{x}, f(\bar{x}))$. Let $f : \mathbb{R}^n \rightarrow (-\infty, +\infty]$ and let $\bar{x} \in \text{dom } f$. The *proximal subdifferential* of f at \bar{x} is defined by

$$\partial^P f(\bar{x}) = \{v \in \mathbb{R}^n : (v, -1) \in N_{\text{epi } f}^P((\bar{x}, f(\bar{x})))\}. \quad (13)$$

The *Fréchet* (resp. *limiting*, *Clarke*) *subdifferential*, denoted $\hat{\partial}f(\bar{x})$ (resp. $\partial f(\bar{x})$, $\partial^C f(\bar{x})$), is defined analogously by replacing normal cone $N_{\text{epi } f}^P(\bar{w})$ with $\hat{N}_{\text{epi } f}(\bar{w})$ (resp. $N_{\text{epi } f}(\bar{w})$, $N_{\text{epi } f}^C(\bar{w})$) in (13) where $\bar{w} = (\bar{x}, f(\bar{x}))$. The *horizon* and *Clarke horizon subdifferentials* at \bar{x} are defined, respectively, by

$$\begin{aligned} \partial_\infty f(\bar{x}) &= \{v \in \mathbb{R}^n : (v, 0) \in N_{\text{epi } f}((\bar{x}, f(\bar{x})))\}, \\ \partial_\infty^C f(\bar{x}) &= \{v \in \mathbb{R}^n : (v, 0) \in N_{\text{epi } f}^C((\bar{x}, f(\bar{x})))\}. \end{aligned}$$

In what follows, we define regularity of functions in terms of the regularity of their epigraphs. We refer to a regularity defined in such a way as *epi-regularity*.

Definition 4.1 (epi-regular functions). *Let $f : \mathbb{R}^n \rightarrow (-\infty, +\infty]$, $\bar{x} \in \text{dom } f$, $\Lambda \subseteq \text{dom } f$, and $(\bar{y}, \bar{v}) \in \text{gph } \partial f \cup \text{gph } \partial_\infty f$.*

- (i) *f is said to be ε -epi-subregular at $\bar{x} \in \text{dom } f$ relative to $\Lambda \subseteq \text{dom } f$ for (\bar{y}, \bar{v}) whenever $\text{epi } f$ is ε -subregular at $(\bar{x}, f(\bar{x}))$ relative to $\{(x, \alpha) \in \text{epi } f \mid x \in \Lambda\}$ for $((\bar{y}, f(\bar{y})), (\bar{v}, e))$ with $e = -1$ when $\bar{v} \in \partial f(\bar{y})$ and $e = 0$ when $\bar{v} \in \partial_\infty f(\bar{y})$.*
- (ii) *f is said to be epi-subregular at \bar{x} relative to $\Lambda \subseteq \text{dom } f$ for (\bar{y}, \bar{v}) whenever $\text{epi } f$ is subregular at $(\bar{x}, f(\bar{x}))$ relative to $\{(x, \alpha) \in \text{epi } f \mid x \in \Lambda\}$ for $((\bar{y}, f(\bar{y})), (\bar{v}, e))$ with $e = -1$ when $\bar{v} \in \partial f(\bar{y})$ and $e = 0$ when $\bar{v} \in \partial_\infty f(\bar{y})$.*
- (iii) *f is said to be epi-Clarke regular at \bar{x} whenever $\text{epi } f$ is Clarke regular at $(\bar{x}, f(\bar{x}))$. Similarly, the function is said to be epi-Clarke super-regular (resp. epi-super-regular, epi-prox-regular) at \bar{x} whenever its epigraph is Clarke super-regular (resp. super-regular, or prox-regular) at $(\bar{x}, f(\bar{x}))$.*

Recent work [2, 4] makes use of the *directional* regularity (in particular Lipschitz regularity) of functions or their gradients. The next example illustrates how this fits naturally into our framework.

Example 4.2. The negative absolute value function $f(x) = -|x|$ is the classroom example of a function that is not Clarke regular at $x = 0$. It is, however, epi-subregular relative to \mathbb{R}_+ at $x = 0$ for the subgradient $v = -1$ with $\varepsilon = 0$ on the neighborhood $U = \mathbb{R}$ for the same reason that the Pac-Man of Example 2.7 is subregular relative to A at the origin for elements in the graph of the A -restricted proximal normal cone. Simply stated, the function looks convex – it’s linear – when restricted to the positive orthant. In contrast, the function f is not ε -epi-subregular at $x = 0$ relative to \mathbb{R} , again, for the same reason why the Pac-Man of Example 2.7 is not ε -subregular at the origin relative to \mathbb{R}^2 . \diamond

In a subsequent section, we develop an equivalent, though more elementary, characterizations of these regularities of functions defined in Definition 4.1.

4.1 Lipschitz continuous functions

In this section, we consider the class of locally Lipschitz functions, which allows us to avoid the horizon subdifferential (since this is always $\{0\}$ for Lipschitz functions). Recall that a set Ω is called *epi-Lipschitz* at $\bar{w} \in \Omega$ if it can be represented near \bar{w} as the epigraph of a Lipschitz continuous function. Such a function is called a *locally Lipschitz representation* of Ω at \bar{w} .

The following notion of *approximately convex* functions was introduced by Ngai, Luc and Thera [12] and turns out to fit naturally within our framework.

Definition 4.3 (approximate convexity). *A function $f : \mathbb{R}^n \rightarrow (-\infty, +\infty]$ is said to be approximately convex at $\bar{x} \in \mathbb{R}^n$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$(\forall x, y \in \mathbb{B}_\delta(\bar{x}))(\forall t \in]0, 1[) : \\ f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) + \varepsilon t(1 - t)\|x - y\|.$$

Daniilidis and Georgiev [5] and subsequently Aussel, Daniilidis and Thibault [1, Theorem 4.14] showed the connection between approximately convex functions and subsmooth sets. Using our results in the previous section, we are able to provide the following extension of their characterization. In what follows, set $\omega = (x, t) \in \mathbb{R}^n \times \mathbb{R}$ and denote by $\pi(\omega) = x$ its projection onto \mathbb{R}^n .

Proposition 4.4 (subsmoothness of Lipschitz epigraphs). *Let Ω be an epi-Lipschitz subset of \mathbb{R}^{n+1} and let $\bar{w} \in \text{bdry}\Omega$. Then the following statements are equivalent:*

- (i) Ω is Clarke super-regular at \bar{w} .
- (ii) Ω is subsmooth at \bar{w} .
- (iii) every locally Lipschitz representation f of Ω at \bar{w} is approximately convex at $\pi(\bar{w})$.
- (iv) some locally Lipschitz representation f of Ω at \bar{w} is approximately convex at $\pi(\bar{w})$.

Proof. The equivalence of (i) and (ii) follows from Theorem 3.4(i). This equivalence does not require Ω to be epi-Lipschitz. The equivalence of (ii), (iii) and (iv) was established in [1, Theorem 4.14]. \square

Remark 4.5. The equivalences in Proposition 4.4 actually hold in the Hilbert space setting without any changes. In fact, the equivalence of (ii)-(iv) remains true in Banach spaces [1, Theorem 4.14]. \diamond

The following characterization extends [5, Theorem 2].

Theorem 4.6 (characterizations of approximate convexity). *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be locally Lipschitz on \mathbb{R}^n and let $\bar{x} \in \mathbb{R}^n$. Then the following are equivalent.*

- (i) $\text{epi } f$ is Clarke super-regular at $(\bar{x}, f(\bar{x}))$.
- (ii) f is approximately convex at \bar{x} .
- (iii) For every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$(\forall x, y \in \mathbb{B}_\delta(\bar{x}))(\forall v \in \partial^{\text{C}}f(x)) \quad f(y) - f(x) \geq \langle v, y - x \rangle - \varepsilon \|y - x\|.$$

- (iv) ∂f is submonotone [5, Definition 7] at x_0 , that is, for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x_1, x_2 \in \mathbb{B}_\delta(x_0) \cap \text{dom } \partial f$, and all $x_i^* \in \partial f(x_i)$ ($i = 1, 2$), one has

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\varepsilon \|x_1 - x_2\|. \quad (14)$$

Proof. (i) \iff (ii): Since f is locally Lipschitz at \bar{x} , it is trivially a local Lipschitz representation of $\Omega = \text{epi } f$ at $\bar{\omega} = (\bar{x}, f(\bar{x})) \in \Omega$. The result thus follows from Proposition 4.4. (ii) \iff (iii) \iff (iv): This is [5, Theorem 2]. \square

As corollary of the above theorem, we deduce the following sufficient condition for Clarke super-regularity of the epigraph.

Corollary 4.7. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be lower semicontinuous (lsc) and approximately convex. Then $\text{epi } f$ is Clarke super-regular at every $(\bar{x}, f(\bar{x})) \in \text{epi } f$.*

Proof. As a proper, lsc, approximately convex function is locally Lipschitz at each point in the interior of its domain [12, Proposition 3.2] and $\text{dom } f = \mathbb{R}^n$, the result follows from Theorem 4.6. \square

Example 4.8 (Clarke super-regularity does not imply approximate convexity). Consider the counting function $f : \mathbb{R}^n \rightarrow \{0, 1, \dots, n\}$ defined by

$$f(x) = \|x\|_0 := \sum_{j=1}^n |\text{sign}(x_j)|, \quad \text{where } \text{sign}(t) := \begin{cases} -1, & \text{for } t < 0 \\ 0, & \text{for } t = 0 \\ +1, & \text{for } t > 0. \end{cases}$$

This function is lower semicontinuous and Clarke epi-super-regular almost everywhere, but not locally Lipschitz at x whenever $\|x\|_0 < n$. In fact, f is actually discontinuous at all such points. Indeed, the epigraph of f is *locally convex* almost everywhere and, in particular, at any point (x, α) with $\alpha > f(x)$. At the point $(x, f(x))$ however, the

epigraph is not even Clarke regular when $\|x\|_0 < n$. Conversely, if x is any point with $\|x\|_0 = n$, then the counting function is locally constant and so in fact locally convex. These observations agree nicely with those in [9], namely, that the rank function (a generalization of the counting function) is subdifferentially regular everywhere (*i.e.*, all the various subdifferentials coincide) with $0 \in \partial\|x\|_0$ for all $x \in \mathbb{R}^n$. \diamond

4.2 Non-Lipschitzian functions

In this section, we collect results which hold true without assuming local Lipschitz continuity. In order to state the following proposition, recall that an extended real-valued function f is strongly amenable at \bar{x} if $f(\bar{x})$ is finite and there exists an open neighborhood U of \bar{x} on which f has a representation as a composite $g \circ F$ with F of class \mathcal{C}^2 and g a proper, lsc, convex function on \mathbb{R}^n .

Proposition 4.9. *Let $f : \mathbb{R}^n \rightarrow (-\infty, +\infty]$ and consider the following assertions.*

- (i) f is strongly amenable at \bar{x} .
- (ii) f is prox-regular at \bar{x} .
- (iii) $\text{epi } f$ is Clarke super-regular at $(\bar{x}, f(\bar{x}))$.

Then: (i) \implies (ii) \implies (iii).

Proof. The fact that strong amenability implies prox-regularity is discussed in [13, Proposition 2.5]. To see that (ii) implies (iii), suppose f is prox-regular at \bar{x} . Then $\text{epi } f$ is prox-regular at $(\bar{x}, f(\bar{x}))$ by [13, Theorem 3.5] and hence Clarke super-regular at $(\bar{x}, f(\bar{x}))$ by Theorem 3.4. \square

To conclude, we establish a *primal* characterization of epi-subregularity analogous to the characterization of Clarke epi-super-regularity in Theorem 4.6. It is worth noting that, unlike the results in Section 4.1, this characterization includes the possibility of horizon normals. In what follows, we denote the epigraph of a function f restricted to a subset $\Lambda \subset \text{dom } f$ by $\text{epi}(f_\Lambda) := \{(x, \alpha) \in \text{epi } f \mid x \in \Lambda\}$.

Proposition 4.10. *Consider a function $f : \mathbb{R}^n \rightarrow (-\infty, +\infty]$, let $\bar{x} \in \text{dom } f$ and let $(\bar{x}, \bar{v}) \in \text{gph } \partial^{\text{C}}f \cup \text{gph } \partial^{\text{C}}_\infty f$. Then the following assertions hold.*

- (i) f has an ε -subregular epigraph at $\bar{x} \in \text{dom } f$ relative to $\Lambda \subseteq \text{dom } f$ for (\bar{x}, \bar{v}) if and only if there is a neighborhood U of $(\bar{x}, f(\bar{x}))$ such that, for all $(x, \alpha) \in \text{epi}(f_\Lambda) \cap U$, one of the following two inequalities holds:

$$f(\bar{x}) + \langle \bar{v}, x - \bar{x} \rangle \leq \alpha + \varepsilon \|\bar{v}\| \|x - \bar{x}\| \left((1 + \|\bar{v}\|^{-2}) (1 + |\alpha - f(\bar{x})|^2 \|x - \bar{x}\|^{-2}) \right)^{1/2}, \quad (15a)$$

$$\langle \bar{v}, x - \bar{x} \rangle \leq \varepsilon \|\bar{v}\| \|x - \bar{x}\| (1 + |\alpha - f(\bar{x})|^2 \|x - \bar{x}\|^{-2})^{1/2}. \quad (15b)$$

(ii) f is epi-subregular at $\bar{x} \in \text{dom } f$ for (\bar{x}, \bar{v}) relative to $\Lambda \subseteq \text{dom } f$ if and only if for all $\varepsilon > 0$ there is a neighborhood (depending on ε) of $(\bar{x}, f(\bar{x}))$ such that, for all $(x, \alpha) \in \text{epi}(f_\Lambda) \cap U$, either (15a) or (15b) holds.

Proof. (i): First observe that since

$$\begin{aligned} N_{\text{epi } f}(\bar{x}) &\supseteq \{(v, -1) \mid v \in \partial f(\bar{x})\} \cup \{(v, 0) \mid v \in \partial_\infty f(\bar{x})\}, \text{ and} \\ N_{\text{epi } f}^{\text{C}}(\bar{x}) &\supseteq \{(v, -1) \mid v \in \partial^{\text{C}} f(\bar{x})\} \cup \{(v, 0) \mid v \in \partial_\infty^{\text{C}} f(\bar{x})\}, \end{aligned}$$

any point $(\bar{x}, \bar{v}) \in \text{gph } \partial f \cup \text{gph } \partial_\infty f$ corresponds to either a normal vector of the form $(\bar{v}, -1)$ or a horizon normal of the form $(\bar{v}, 0)$. Suppose first that f is ε -epi-subregular at \bar{x} relative to $\Lambda \subset \text{dom } f$ for $(\bar{v}, \bar{x}) \in \text{gph } \partial^{\text{C}} f$ with neighborhood U' of \bar{x} . Then epi f is ε -subregular at $(\bar{x}, f(\bar{x}))$ relative to $\text{epi}(f_\Lambda)$ for $(\bar{v}, -1) \in N_{\text{epi } f}^{\text{C}}(\bar{x}, f(\bar{x}))$ with constant ε and neighborhood U of $(\bar{x}, f(\bar{x}))$ in (3). Thus, for all $(x, \alpha) \in \text{epi}(f_\Lambda) \cap U$, we have

$$\begin{aligned} \langle (\bar{v}, -1), (x, \alpha) - (\bar{x}, f(\bar{x})) \rangle &\leq \varepsilon \|(\bar{v}, -1)\| \|(x, \alpha) - (\bar{x}, f(\bar{x}))\| \\ \iff \langle \bar{v}, x - \bar{x} \rangle - \alpha + f(\bar{x}) &\leq \varepsilon (\|\bar{v}\|^2 + 1)^{1/2} (\|x - \bar{x}\|^2 + (\alpha - f(\bar{x}))^2)^{1/2} \\ &= \varepsilon \|\bar{v}\| \|x - \bar{x}\| (1 + \|\bar{v}\|^{-2})^{1/2} (1 + (\alpha - f(\bar{x}))^2 \|x - \bar{x}\|^{-2})^{1/2}, \end{aligned}$$

from which the claim follows.

The only other case to consider is that f is ε -epi-subregular at \bar{x} relative to $\Lambda \subset \text{dom } f$ for $(\bar{v}, 0) \in \partial_\infty^{\text{C}} f(\bar{x})$ with constant ε and neighborhood U' of \bar{x} . In this case, epi f is ε -subregular at $(\bar{x}, f(\bar{x}))$ relative to $\text{epi}(f_\Lambda)$ for $(\bar{v}, 0) \in N_{\text{epi } f}^{\text{C}}(\bar{x}, f(\bar{x}))$ with constant ε and neighborhood U of $(\bar{x}, f(\bar{x}))$ in (3). Thus, for all $(x, \alpha) \in \text{epi}(f_\Lambda) \cap U$, we have

$$\begin{aligned} \langle (\bar{v}, 0), (x, \alpha) - (\bar{x}, f(\bar{x})) \rangle &\leq \varepsilon \|(\bar{v}, 0)\| \|(x, \alpha) - (\bar{x}, f(\bar{x}))\| \\ \iff \langle \bar{v}, x - \bar{x} \rangle &\leq \varepsilon \|\bar{v}\| (\|x - \bar{x}\|^2 + (\alpha - f(\bar{x}))^2)^{1/2} \\ \iff \langle \bar{v}, x - \bar{x} \rangle &\leq \varepsilon \|\bar{v}\| \|x - \bar{x}\| (1 + (\alpha - f(\bar{x}))^2 \|x - \bar{x}\|^{-2})^{1/2}, \end{aligned}$$

which completes the proof of (i).

(ii): Follows immediately from the definition. \square

Remark 4.11 (indicator functions of subregular sets). When $f = \iota_\Omega$ for a closed set Ω the various subdifferentials coincide with the respective normal cones to Ω . In this case, inequality (15b) subsumes (15a) since all subgradients are also horizon subgradients and (15b) reduces to (3) in agreement with the corresponding notions of regularity of sets. \diamond

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