Tiling with bars under tomographic constraints

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Abstract

We wish to tile a rectangle or a torus with only vertical and horizontal bars of a given length, such that the number of bars in every column and row equals given numbers. We present results for particular instances and for a more general problem, while leaving open the initial problem.

Keywords: Discrete Tomography, Domino Tiling

1   Introduction

In general terms, *tomography* is the area of reconstructing objects from lower dimensional projections. We consider the problem of reconstructing a rectangular grid from projections on the columns and on the rows. Think of the grid as a layer in a crystal, and in order to measure it, we send beams through the crystal from two orthogonal directions. Measurements will give us quantitative information about columns or rows of the grid (3). Consider the problem, where each cell of the grid (think of it as an atom) is matched with at most one of its immediate neighbors (think of it as a chemical connection). Physicists call them *monomer-dimer systems*. Many research has been done about counting the number of configurations of such a system (2), or about almost uniform randomly sampling configurations (4).

We are interested in the particular problem, were each cell is matched to exactly one neighboring cell. These objects correspond to domino tilings of the grid. A measurement will reveal the number of vertical dominoes in each column and the number of horizontal dominoes in each row. Given these numbers we wish to reconstruct the grid, or any grid which satisfies the projection constraints. As a natural generalization of this combinatorial problem we are interested in the tiling of the grid with horizontal bars of length h and vertical bars of length v, for some integers h, v. We call it the TILING WITH BARS RECONSTRUCTION problem. Given a pair of column and row vectors $(\underline{m}, \underline{n})$ (*tomographic constraints*) and integers h, v we want to construct *a* tiling with bars, such that $\underline{m}$ counts the number of vertical bars in the columns and $\underline{n}$ counts the number of horizontal bars in the rows. This problem has two variants, whether we tile a rectangle or a torus.

Figure 1: A tiling and its projections.

The problem is left open by this paper, but we were able to find solutions for sub-problems and for a more general problem. We summarize our results in this table:
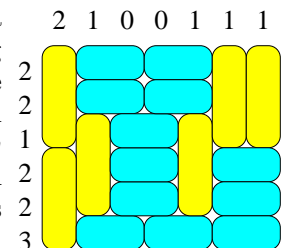
	rectangle	torus
tiling of a given sub-grid	NP-hard	NP-hard
general problem	open	open
$\underline{m}$ is a uniform vector	quadratic algorithm	open
$\underline{n}$ is also a uniform vector	algebraic characterization	algebraic characterization

The quadratic algorithm has been found independently by Picouleau (5) for a more general condition (see end of section 4.3).

2 Definitions

Let $a, b \geq 1$. The rectangle $R_{a \times b}$ is the product $\{0, \cdots, a-1\} \times \{0, \cdots, b-1\}$ and the torus $T_{a \times b}$ is the product $\mathbb{Z}_a \times \mathbb{Z}_b$. Columns are numbered from left to right and rows from top to bottom (see Figure 2). Each element of a rectangle or a torus is called a *cell*.
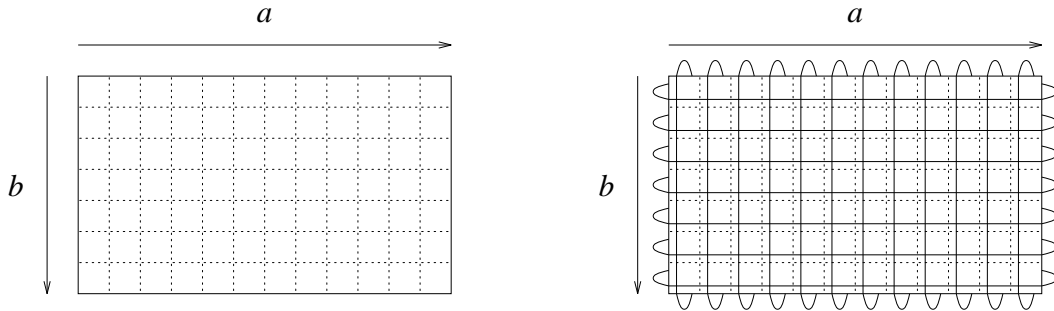

Figure 2: A rectangle $R_{a \times b}$ and a torus $T_{a \times b}$.

Let $h, v \geq 1$. The horizontal bar of length h is the rectangle $R_{h \times 1}$ and the vertical bar of length v is the rectangle $R_{1 \times v}$. If the length is 2 we call the bar a *domino*.

A rectangle $R_{a \times b}$ (respectively a torus $T_{a \times b}$) is said to be *tillable* with the vertical and horizontal bars (of lengths v and h respectively) if it can be partitioned into those bars. The projections of such a tiling is the pair of vectors $(\underline{m}, \underline{n}) = (m_1 \cdots m_a, n_1 \cdots n_b) \in \mathbb{N}^a \times \mathbb{N}^b$ such that for every column i, m_i is the number of vertical bars in it, and for every row j, n_j is the number of horizontal bars in it.

We define the following reconstruction problems.

TILING A RECTANGLE (RESPECTIVELY TORUS) WITH BARS UNDER TOMOGRAPHIC CONSTRAINTS

input $(\underline{m}, \underline{n}) \in \mathbb{N}^a \times \mathbb{N}^b$ and $h, v \geq 1$.

output a tiling of the $R_{a \times b}$ (respectively $T_{a \times b}$) with projections $(\underline{m}, \underline{n})$.

3 Uniform constraints

In this section we characterize valid instances for the special case when the constraints vectors are uniform, that is $\forall i : m_i = m, \forall j : n_j = n$ for some integers m, n. Both the torus and the rectangle case are studied.

3.1 The torus case

The number of cells covered with vertical bars is amv and the number of cells tiled with horizontal bars bnh. Clearly these numbers must add up to the total number of cells ab. In this section we show that this condition is sufficient for a torus tiling to exist.

Lemma 1 *If $a, b, h, v, m, n \geq 1$ are such that $ab = amv + bnh$ then there exist $p, q, a', b' \geq 1$ satisfying*

- $(p + q) = \gcd(a, b)$ *and* $a = (p + q)a'$ *and* $b = (p + q)b'$.

- $nh = pa'$ *and* $mv = qb'$.

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Proof. If we denote $c = \gcd(a, b)$, $a = ca'$, and $b = cb'$, then the equality $ab = amv + bnh$ can be rewritten as $ca'b' = a'mv + b'nh$. It follows that $(ca' - nh)b' = a'mv$. From Gauss theorem, $a'|(ca' - nh)$, and therefore $a'|nh$. In other words, there exists p such that $pa' = nh$. By symmetry, there exists q such that $qb' = mv$. Finally, notice that

$$p + q = \frac{c(cb'nh + ca'mv)}{c^2 a'b'} = c.$$

$\square$

Theorem 1 *Let $a, b, h, v, m, n \geq 1$. Let $(\underline{m}, \underline{n}) = (m \cdots m, n \cdots n) \in \mathbb{N}^a \times \mathbb{N}^b$. The torus $T_{a \times b}$ is $(\underline{m}, \underline{n})$-tillable with the bars $R_{h \times 1}$ and $R_{1 \times v}$ if and only if $ab = amv + bnh$.*

Proof. If we assume that the torus $T_{a \times b}$ admits an $(\underline{m}, \underline{n})$-tiling with the bars $R_{h \times 1}$ and $R_{1 \times v}$ then, by simple considering the area covered by the tiling, it is direct to notice that $ab = amv + bnh$.

Conversely, if $ab = amv + bnh$ then, by Lemma 1, there exist $p, q, a', b' \geq 1$ such that:

- $(p + q) = \gcd(a, b)$ and $a = (p + q)a'$ and $b = (p + q)b'$.

- $nh = pa'$ and $mv = qb'$.

As it appears in Figure 3, the torus $T_{a \times b}$ can be partitioned into $(p + q)^2$ rectangles $\Theta_{i,j}$ defined for each $i, j \in \{0, \cdots, p + q - 1\}$ as follows:

$$\Theta_{i,j} = \text{the rectangle } R_{a' \times b'} \text{ whose upper left corner is the cell } (a'i, b'j).$$

Let us define for each $i \in \{0, \cdots, p + q - 1\}$, the following rectangular regions of $T_{a \times b}$:

- $\mathcal{H}_i = \bigcup_{k=i}^{i+p-1} \Theta_{k,i}$, which is simply the rectangle $R_{nh \times b'}$ whose upper left corner is the cell $(a'i, b'i)$,

- $\mathcal{V}_i = \bigcup_{k=i+1}^{i+q} \Theta_{i,k}$, which is simply the rectangle $R_{a' \times mv}$ whose upper left corner is the cell $(a'i, b'(i+1))$.

It is easy to notice that every $\Theta_{i,j}$ belongs to exactly one of the rectangles $\{\mathcal{H}_i, \mathcal{V}_i\}_{0 \leq i < p+q}$ and that therefore the latter is a partition of the torus $T_{a \times b}$.

In order to conclude, notice that each $\mathcal{H}_i$ is tillable by using only horizontal bars $R_{h \times 1}$ with each row having n bars. In the same way, each $\mathcal{V}_j$ is tillable by using only vertical bars $R_{1 \times v}$ with each column having m bars. $\square$
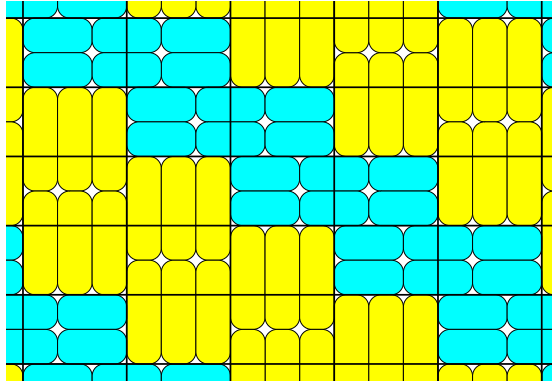

Figure 3: A $(2 \cdots 2, 3 \cdots 3)$-tiling of $T_{15 \times 10}$ by $R_{2 \times 1}$ and $R_{1 \times 3}$.

Corollary 1 *If a torus $T_{a \times b}$ admits a tiling with uniform tomographic constraints then $\gcd(a, b) > 1$.*

Proof. From Theorem 1 together with Lemma 1, $\gcd(a, b) = p + q$ with $p, q \geq 1$. $\square$

3.2 The rectangle case

Theorem 2 *Let $a, b, h, v, m, n \geq 1$. Let $(\underline{m}, \underline{n}) = (m \cdots m, n \cdots n) \in \mathbb{N}^a \times \mathbb{N}^b$. The rectangle $R_{a \times b}$ is $(\underline{m}, \underline{n})$-tillable with the bars $R_{h \times 1}$ and $R_{1 \times v}$ if and only if $ab = amv + bnh$, $h|a$, and $v|b$.*

Proof. Let us suppose that the rectangle $R_{a \times b}$ admits an $(\underline{m}, \underline{n})$-tiling with the bars $R_{h \times 1}$ and $R_{1 \times v}$. By simple area considerations, it holds that $ab = amv + bnh$. The fact $h|a$ follows from this observation. Since in every column we have mv cells tiled by vertical bars, the remaining $k = b - mv$ cells are tiled with horizontal bars. Therefore k horizontal bars are between column 1 and column h, another k bars between columns $h+1$ and $2h$, and so on. In the same way we conclude $v|b$.

For the converse, let $ab = amv + bnh$, $h|a$, and $v|b$. We reduce this case to a 01-MATRIX RECONSTRUCTION PROBLEM. Let p, q be such that $a = ph$ and $b = qv$. Now $R_{a \times b}$ may be partitioned into pq rectangles $R_{h \times v}$ and each of these rectangles may be tiled by using one type of bars (vertical or horizontal). We define a $p \times q$ 01-matrix, where each entry corresponds to a $R_{h \times v}$ rectangle, and contains "1" if the later is tiled with vertical bars, and "0" otherwise. The problem is reduced to the following: given $p, q, m, n \geq 1$ such that $pq = pm + qn$, construct a 0-1 matrix of size $p \times q$ in such a way that each column has m 1's and each row has n 0's. The solution is trivial. In fact, it suffices to consider the 1's as vertical bars of unitary length (simple squares), the 0's as horizontal bars of unitary length (simple squares), and to apply Theorem 1 (for unitary length bars the torus is equivalent to the rectangle). $\square$

4 Horizontal bars of unit length

In this section, we give a polynomial time algorithm for reconstructing a rectangle tiling with horizontal bars of unit length. We assume in this section that $h = 1$. For technical reasons we will even give a more general algorithm for reconstructing tilings of *histograms*.

Definition 1 *A histogram H of a rectangle $R_{a \times b}$ is a subset of $R_{a \times b}$ such that if cell (i, j) is an element of H with $j < b - 1$, then $(i, j+1)$ is also an element of H. The* top *of column i is the cell $(i, j) \in H$ with minimal j (remember rows are numbered from top to bottom). The number of cells of the row j of H is denoted by c_j. The height of column i of H is the number of cells in it.*

We will give an algorithm which, given a vector $(m_0, m_1, \ldots, m_{a-1}, n_0, n_1, \ldots, n_{b-1})$ of integer coordinates, an integer v and a histogram H of rectangle $R_{a \times b}$, constructs a tiling of H with the bars $R_{1 \times 1}$ and $R_{1 \times v}$ satisfying the tomographic constraints $(\underline{m}, \underline{n})$ or answers "No" if there is no such tiling.

4.1 Algorithm

This algorithm is based on a very simple idea: A solution is constructed iteratively row by row, where vertical tiles are placed in columns of largest remaining constraint. See *http://www.lri.fr/˜durr/VertOnly/-vertOnly.html* for an implementation.

input: $\underline{m} \in \mathbb{N}^a, \underline{n} \in \mathbb{N}^b, v > 0$, histogram $H \subseteq R_{a \times b}$.
promise: $\sum_{i=0}^{a-1} vm_i + \sum_{j=0}^{b-1} n_j = |H|$.
For j^* from 0 to $b - 1$ **do**
    If $c_{j^*} < n_{j^*}$ answer "No" and stop.
    While $c_{j^*} > n_{j^*}$ do
        If $j^* + v - 1 \geq b - 1$ answer "No" and stop.
        Let i be a column with maximal m_i which satisfies $(i, j^*) \in H$.
        Place a vertical bar between (i, j^*) and $(i, j^* + v - 1)$.
        For k from j^* to $j^* + v - 1$ **do**
            Remove cell (i, k) from H.
            Update $c_k := c_k - 1$.
        Update $m_i := m_i - 1$.
    While $n_{j^*} > 0$ do

Let i be any column which satisfies $(i, j^*) \in H$.
Place a horizontal bar (cell actually) on (i, j^*).
Remove this cell from H.
Update $n_{j^*} := n_{j^*} - 1$.

4.2 Analysis

The correctness of the algorithm is a consequence of the lemma below:

Lemma 2 *Let T be a tiling of an histogram H satisfying the constraints given by vector $(m_0, \ldots, m_{a-1}, n_0, \ldots, n_{b-1})$. Let S_{algo} be the set of columns of maximal height where vertical bars are placed in the first step of the algorithm.*

Assume that there exists a solution. For each tiling T which solves our problem, let S_T be the set of columns with maximal height in H whose top is covered by a vertical bar. Let T^* be such a solution such that $S_{algo} \cap S_{T^*}$ is maximal. Then $S_{algo} = S_{T^*}$

Proof by contradiction. Since $|S_{algo}| = |S_{T^*}|$ there is a column i_1 of S_{algo} which is not an element of S_{T^*}, and a column i_2 of S_{T^*} which is not an element of S_{algo}. Fix such a pair of columns i_1, i_2. Notice that $m_{i_2} \leq m_{i_1}$.

Let V_1 (respectively V_2) denote the set of vertical bars of T^* included in column i_1 (respectively column i_2). We construct a bipartite undirected graph G whose set of vertices is $V_1 \cup V_2$. Two tiles are joined by an edge if they cross a same row. Notice that, in G, each bar has at most two neighbors, and G has no cycle. Hence, G is formed from disjoint chains (each bar with no neighbor is considered as a chain of null length). For every chain C let I_C be the set of rows traversing the bars in C. Then clearly I_C is an interval, and different chains have disjoint row sets.

From such a chain C, one can construct a tiling T_C transforming T^*, by an *exchange on chains*, which is the exchange of $I_C \times \{i_1\}$ with $I_C \times \{i_2\}$. Notice that this operation preserves the tomographic constraints on the rows, while preserving those on the columns if and only if C has an even number of vertices.

Let C_0 be the chain with lowest row indices.

If C_0 has an even number of vertices, then the tiling T_{C_0} contradicts the maximality of the intersection $S_{algo} \cap S_{T^*}$, which, consequently, achieves the proof.

Figure 4: The exchange of $I_C \times \{i_1\}$ with $I_C \times \{i_2\}$.

If C_0 has an odd number of vertices (i.e. both endpoints of C_0 are bars in column i_2), then by the inequality $m_{i_2} \leq m_{i_1}$ there exists another chain C_1 with an odd number of vertices, whose extremities are bars in column i_1. Let T' be the tiling obtained from T^* by exchanging C_0 and C_1. T' satisfies the same vertical and horizontal constraints as T^*, and, consequently contradicts the maximality of the intersection $S_{algo} \cap S_{T^*}$. This last fact achieves the proof.
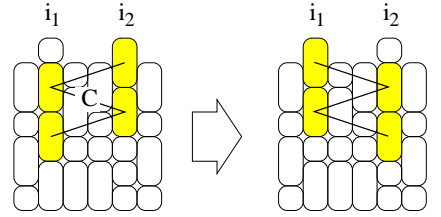
$\square$

Lemma 3 *The algorithm presented in this section gives a tiling satisfying the constraints, if such a tiling exists. It's running time is $O(a \log a + ab)$.*

Proof. We prove its correctness by induction on the number of cells of the histogram H given as input. If H is empty, the result is obvious.

Now assume that the theorem holds for each histogram which has less cells than H. If H admits a tiling with constraints, then, by the previous lemma, there exists such a tiling using tiles placed in the first execution of the loop.

After the first execution of the loop (and updating), we have to prove the theorem for an histogram which has less cells than H, which is true by induction hypothesis.

Now we turn to the proof of the time complexity. The algorithm will maintain an ordering on $\underline{n}$.

- the initialization costs $O(a \log a)$ time units (because of the ordering of the columns),

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Figure 5: Trace of the reconstructing algorithm on Example 1.

- each passage through the loop costs $O(a)$ time units, since the update of the order can easily be done in $O(a)$ time units,

- there are b passages through the loop.

This proves that the execution of this algorithm costs $O(a \log a + ab)$ time units. $\qquad\square$

Example 1 *For the rectangle $R_{10 \times 8}$, for vertical bars of size 2 and the projections*

$$\underline{m} = (2, 3, 3, 1, 3, 3, 4, 3, 2, 2) \in \mathbb{N}^{10} \ \ and \ \underline{n} = (4, 2, 3, 3, 4, 3, 2, 6) \in \mathbb{N}^8.$$

Figure 5 shows the trace of the algorithm. Numbers in the cells indicate the remaining column constraints.

4.3 Application

The previous algorithm can be used to reconstruct a tiling, when some particular promise on $\underline{m}$ is given. This promise is fulfilled in particular if $\underline{m}$ is uniform, or monotone ($m_0 \leq \ldots \leq m_{a-1}$), as shown in (5).

Theorem 3 *Let $a, b, h, v \geq 1$. Let $(\underline{m}, \underline{n}) = (m_0 \cdots m_{a-1}, n_0 \cdots n_{b-1}) \in \mathbb{N}^a \times \mathbb{N}^b$ with $m_i = m_j$ for all $i, j \in \{0, \ldots, a-1\}$ satisfying $\lfloor i/h \rfloor = \lfloor j/h \rfloor$. There is an algorithm in $O(a \log a + ab)$ that decides whether the rectangle $R_{a \times b}$ is $(\underline{m}, \underline{n})$-tillable with the bars $R_{h \times 1}$ and $R_{1 \times v}$ and if yes outputs a valid tiling.*

Proof. By the same argument of the first part of Theorem 2 it can be concluded that the tiling of $R_{a \times b}$ may be partitioned into $\frac{a}{h}$ tilings of rectangles of type $R_{h \times b}$. It suffices now to divide every horizontal measure by h (i.e, to change the horizontal scale) in order to reduce the original problem to a new one in which $a' = \frac{a}{h}$, $b' = b$, $h' = \frac{h}{h} = 1$, $v' = v$, $(\underline{m})' = (m_0, m_h, \ldots, m_{(a'-1)h}) \in \mathbb{N}^{a'}$, $(\underline{n})' = \underline{n} \in \mathbb{N}^b$ (see Figure 6). We can apply now Lemma 3. $\qquad\square$

5 Tiling a sub-grid

In the previous section we showed that some instances of the TILING WITH BARS RECONSTRUCTION problem have a polynomial solution. In this section we show that a more general problem is NP-hard. In the SUB-GRID DOMINO TILING RECONSTRUCTION PROBLEM FROM PROJECTIONS we are given a only sub-grid $S \subseteq R_{a \times b}$ to tile. We show that this problem is NP-hard by a reduction from the following problem.

THE 3-COLOR CONSISTENCY PROBLEM
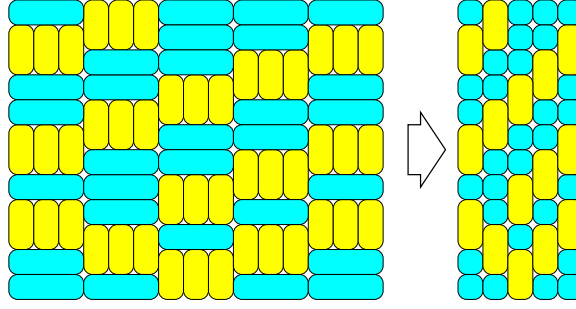We fix a set of colors $\Delta = \{colorless, red, blue, green\}$.

Figure 6: Reducing the problem by changing its horizontal scale.

input $\underline{m}^c \in \mathbb{N}^a$ and $\underline{n}^c \in \mathbb{N}^b$ for every $c \in \Delta$.

decide if there is a matrix $T \in \Delta^{a \times b}$ with projections $(\underline{m}^c, \underline{n}^c)_{c \in \Delta}$ that is for all colors c we have

$$m_i^c = |\{j : T_{ij} = c\}| \text{ and } n_j^c = |\{i : T_{ij} = c\}|.$$

It has been shown in (1) that this problem is NP-hard in the strong sense, while the 1-COLOR CONSISTENCY PROBLEM is solvable in linear time (6). The 2-COLOR CONSISTENCY PROBLEM is still open.

5.1 Gadget

A sub-grid is a *cycle* if every cell has exactly two (horizontal or vertical) adjacent neighbors, and if every pair of cells is connected by transitivity. We start by giving some facts about cycles.

The cycle length is always even. This can be easily seen by coloring the cells checkerboard wise black and white. Then adjacent cells have different colors. The claim follows from the fact that the cycle is closed.

There are exactly two domino tilings of a cycle. Fix any numbering of the cells such that every cell i has neighbors $i - 1$ and $i + 1$ modulo the length of the cycle. Then clearly one tiling covers all pair of cells $(2i, 2i + 1)$ with a domino, while the other one covers $(2i, 2i - 1)$ for all i.

We specify now a sub-grid S consisting of two cycles intersecting at a corner and an addition cell. This additional cell must, in a domino tiling, be matched to a cell of one of the cycles, therefore "forcing" it to admit a unique tiling, while the other one admits the usual two tilings. As a result we will have exactly four tilings of S. We define S to be the subgrid shown with its tilings in figure 7. We refer to these tilings as $T_{colorless}, T_{red}, T_{blue}, T_{green}$ respectively. Let $(\underline{s}^c, \underline{t}^c)$ be their projection vectors for every color c. Note that by the symmetry of S we have $\{\underline{s}^c\}_{c \in \Delta} = \{\underline{t}^c\}_{c \in \Delta}$.
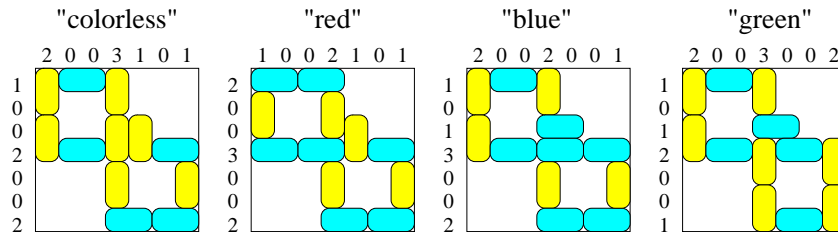

Figure 7: All four tilings of S.

Lemma 4 *The vectors $\{\underline{s}^c\}_{c \in \Delta}$ are linear independent.*

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Proof. Let $\underline{u} \in \mathbb{N}^7$ be an arbitrary linear composition of the column vectors. We have to show that the coefficients in $\underline{u} = \sum_c \alpha_c \underline{s}^c$ are uniquely defined. We have

$$
\begin{aligned}
u_1 &= 2\alpha_{colorless} + 1\alpha_{red} + 2\alpha_{blue} + 2\alpha_{green} \\
u_3 &= 3\alpha_{colorless} + 2\alpha_{red} + 2\alpha_{blue} + 3\alpha_{green} \\
u_4 &= 1\alpha_{colorless} + 1\alpha_{red} + 0\alpha_{blue} + 0\alpha_{green} \\
u_7 &= 1\alpha_{colorless} + 1\alpha_{red} + 1\alpha_{blue} + 2\alpha_{green}
\end{aligned}
$$

This system of equations has a unique solution which concludes the proof: $\alpha_{colorless} = -2, \alpha_{red} = 3, \alpha_{blue} = 2, \alpha_{green} = -1$. $\qquad\square$
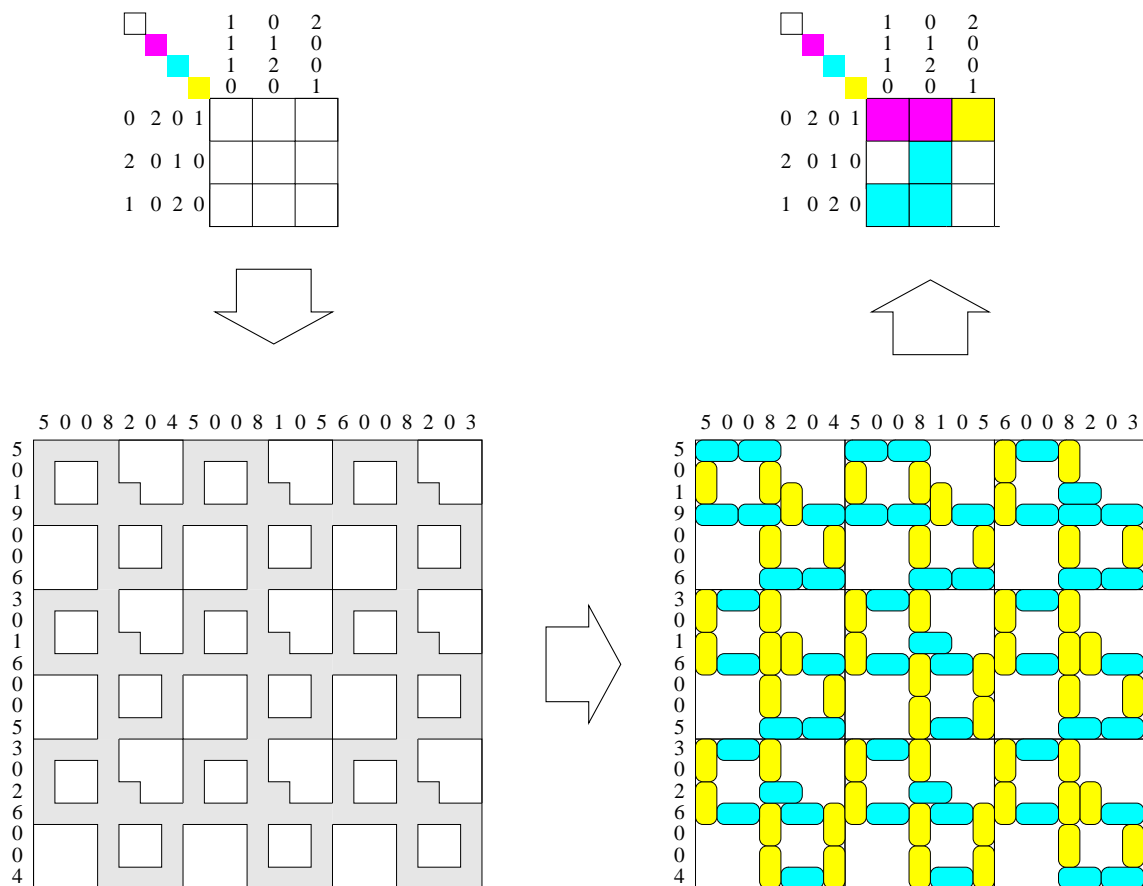
5.2 The proof of NP-hardness

Figure 8: Idea of the reduction

Theorem 4 *The* DOMINO SUB-GRID TILING RECONSTRUCTION PROBLEM FROM PROJECTIONS *is NP-hard in the strong sense.*

Proof. Let $I = (\underline{m}^c, \underline{n}^c)$ an arbitrary instance of the 3-COLOR CONSISTENCY problem for an $a \times b$ matrix. We construct an instance of the former problem, such that there is a bijection between the respective sets of solutions. This proves then the theorem. (see figure 8)

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	2	1	2	2	1	2
1						
0						
2						
2						
1						
2						

	2	1	2	2	1	2	4
3							
3							
1							
0							
2							
2							
1							
2							

Figure 9: Separations for $h = 2, v = 2$.

We define the instance I' as $(\underline{m}, \underline{n}) \in \mathbb{N}^{7a} \times \mathbb{N}^{7b}$ with

$$\underline{m} = \bigotimes_{i=1}^{a} \left(\sum_c m_i^c \underline{s}^c \right) \text{ and } \underline{n} = \bigotimes_{j=1}^{b} \left(\sum_c n_j^c \underline{t}^c \right),$$

where $\otimes$ denotes the concatenation of vectors and

$$S' = \bigcup_{i=0}^{a-1} \bigcup_{j=0}^{b-1} (S + (7i, 7j)).$$

In a tiling of S' every 7×7 block contains one of the four tilings of S. Therefore there is a natural bijection f between the set of domino tilings of S and the set of matrices $\Delta^{a \times b}$. It follows from lemma 4 that T is a solution to the instance I' if and only if $f(T)$ is a solution of I. Moreover for the unary encoding the size of $f(T)$ is linear in the size of T. □

6 Concluding remarks

We will conclude with a observation for the general problem. Let $(\underline{m}, \underline{n}, h, v)$ be an instance for the reconstruction problem for tilings of an $a \times b$-rectangle.

We say that a particular realization is *separated* between column i and column $i + 1$, if there is no horizontal bar traversing the border in between. We claim that if this is the case for one realization, it holds for all other realizations as well: Let c_i be the number of horizontal bars beginning in column i and ending in column $i + h - 1$. Let us also denote $c_i = 0$ for all $i < 0$. Then clearly the following induction holds:

$$c_i = b - (vm_i + c_{i-h+1} + \ldots + c_{i-1}).$$

Therefore there is a separation between column i and column $i+1$ if and only if $c_{i-h+2} + \ldots + c_i = 0$, which is a realization independent condition.

In the same manner we define the separation between lines. These separations, which can be computed in linear time, partition the grid into *separated rectangles* which are surrounded either by a separation or by the border of the grid. Clearly if there is a realization of $(\underline{m}, \underline{n}, h, v)$ then

1. $(v\underline{m}, \underline{n}, h, 1)$ and $(\underline{m}, h\underline{n}, 1, v)$ must have a realization as well

2. and every separated rectangle must admit a tiling with horizontal bars of length h and vertical bars of height v, even without any tomographic constraint.

Left part of figure 9 shows an instance which satisfies the first but not the second condition, since each of the four separated rectangle has odd size. However the two conditions are not sufficient: The right hand instance satisfies conditions 1 and 2. But it has no solution, since the last column must be filled with vertical dominoes, the remaining cells of the first row must be tiled with horizontal dominoes, and for the remaining rectangle we end up with the left hand side instance. However the two conditions are not sufficient, as shows the right part.

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