Tiling Allowing Rotations Only *

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Abstract. *We define a "rotating board" as a finite square with one tile fixed in each cell. These fixed tiles can only be rotated and, in addition, they belong to a particular set of four tiles constructed with two colors. In this paper we show that any set of tiles $\mathcal{T}$ may be coded in linear time into a "rotating board" B in the following sense:*

i. There exists an injection from the colors of the tiles of $\mathcal{T}$ into the set of border conditions of the board B.

ii. There is a one-to-one relation between the set $\mathcal{T}$ and the set of tilings of B (obtained by rotating its tiles) satisfying that each $t \in \mathcal{T}$ is associated to a tiling B_θ in such a way that the (north, south, east and west) colors of t are related to the (north, south, east and west) border conditions of B_θ by the injection of i.

The existence of this coding means that we can efficiently transform an arbitrary degrees of freedom tiling problem (in which to each cell is assigned an arbitrary set of admissible tiles) into a restricted four degrees of freedom problem (in which the tiles, fixed in each cell, can only be rotated). Considering the classical tiling results, we conclude the NP-completeness (resp. undecidability) of the natural bounded (resp. unbounded) version of the rotation tiling problem.

1. Introduction

Wang tiles, or simply tiles, are unit-sized squares with colored edges. The classical translation tiling problem, denoted here by TRANS, was introduced in [Wan61] and since then many different versions of it have been studied: bounded tiling [Lew78,vEB83], recurring tiling [Har86], domino snake [EPH94], rotation tiling [GR97].

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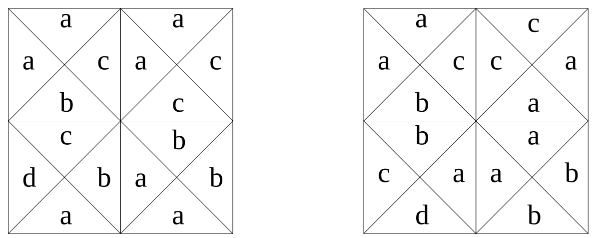
TRANS consists of deciding whether an arbitrary region $R \subseteq \mathbb{Z}^2$ is tilable by a given set of tiles allowing translations only. It is widely known that TRANS is undecidable when $R = \mathbb{Z}^2$ [Be66] and NP-complete when the regions R are finite squares [Lew78].

The rotation tiling problem ROT was used in [Mat91] as a model for the study of some integrated circuit CAD and its complexity was partially studied in [GR97]. In ROT one admissible tile is fixed in each cell of some region R and it is asked whether there exists a tiling when only rotations are allowed (see Figure 1).



(a) (b)

Figure 1. (a) ROT instance or rotating board. **(b)** A tiling obtained by rotations.

Notice that both TRANS and ROT are subproblems of the generalized translation problem GEN-TRANS in which each cell has its own set of admissible tiles (the same for all in TRANS and the four rotations of a given tile in ROT). In this context, if we define the *degrees of freedom* of a cell as the number of admissible tiles associated with it, it follows that TRANS is an arbitrary degrees of freedom problem while ROT is a four degrees of freedom problem.

In [GR97] it is proved that for a particular set of 5 admissible tiles and for square regions, ROT is NP-complete. In this paper we improve and generalize this result by coding arbitrary tile sets in $\mathcal{T}_0$-rotating boards (where $\mathcal{T}_0$ is the set of admissible tiles of Figure 2). For this purpose we represent arbitrary tile sets in the border conditions of the tilings of a $\mathcal{T}_0$-rotating board. In other words, we reduce GEN-TRANS (and therefore TRANS) to ROT.

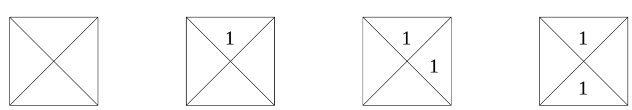


Figure 2. The set of admissible tiles $\mathcal{T}_0$

As a corollary, considering the classical tiling results, one may conclude that:

· ROT is undecidable when the assignments are periodic on the whole plane (using just the set of tiles $\mathcal{T}_0$).

· ROT is NP-complete when the assignments are done on finite squares (using just the set of tiles $\mathcal{T}_0$).

Moreover, ROT inherits any undecidability or complexity result from GEN-TRANS (and therefore also from TRANS).

Definition. Let $R \subseteq \mathbb{Z}^2$ be a region. Let $\mathcal{T}$ be a finite set of tiles. We denote by $T_0 \in \mathcal{T}^R$ an assignment to the region R of tiles belonging to $\mathcal{T}$. T_0 is called a rotating region (rotating board when $R = S_n = \{1, \cdots, n\}^2$, and rotating plane when $R = \mathbb{Z}^2$). We say that $T_0 \in \mathcal{T}^{\mathbb{Z}^2}$ is a periodic tile assignment if there exists $n > 0$ such that $\forall i, j \in \mathbb{Z}$, $T_0(i,j) = T_0(i+n, j) = T_0(i, j+n)$. We denote by T_θ the configuration obtained from T_0 by tile rotations only ($\theta \in \{0°, 90°, 180°, 270°\}^R$).

Definition. PERIOD-ROT($\mathcal{T}_0$).

INSTANCE: $T_0 \in \mathcal{T}_0^{\mathbb{Z}^2}$ periodic tile assignment.
QUESTION: $\exists \theta$ such that T_θ is a tiling?

Definition. BOUND-ROT($\mathcal{T}_0$).

INSTANCE: $T_0 \in \mathcal{T}_0^{S_n}$, where S_n is the $n \times n$ square.
QUESTION: $\exists \theta$ such that T_θ is a tiling?

2. Arbitrary Sets of Tiles and $\mathcal{T}_0$ - Rotating Boards

Definition. We say that a rotating board $B_0 \in {\mathcal{T}_0}^{S_k}$ *codes* an arbitrary set of tiles $\mathcal{T}$ if:

i. There exists an injection from the colors of $\mathcal{T}$ to the set of border conditions $\{\#, 1\}^k$ (where # represents the blank color of the set of tiles $\mathcal{T}_0$).
ii. There is a one-to-one relation between the set of tiles $\mathcal{T}$ and the set of tilings of B_0 satisfying that each $t \in \mathcal{T}$ is associated to a tiling B_θ in such a way that the (north, south, east and west) colors of t are related to the (north, south, east and west) border conditions of B_θ by the injection of i.

The key result of the paper is the following:

Theorem. For every finite set of tiles $\mathcal{T}$ we can construct in linear time a $\mathcal{T}_0$-rotating board B_0 that codes it.

The proof of this theorem will be developed in the next two sections. As in [Ro91], it associates logical circuits to tiling problems. Roughly we code an arbitrary set of tiles $\mathcal{T}$ as a logical circuit lying on a $m \times m$ lattice square (m even), and then we generate B_0 by replacing each circuit cell by a 5×5 $\mathcal{T}_0$-rotating subboard.

As a result of this simulation we obtain a one-to-one relation between $\mathcal{T}$ and the tilings of B_0. More precisely, the idea is to have, for each tiling of B_0, a unique color 1 in each border (in the *positions* of the colors of some tile in $\mathcal{T}$). Notice that the fact that the size of B_0 is even ($5m \times 5m$) allows us to view the square colored like a chessboard (see Figure 3).

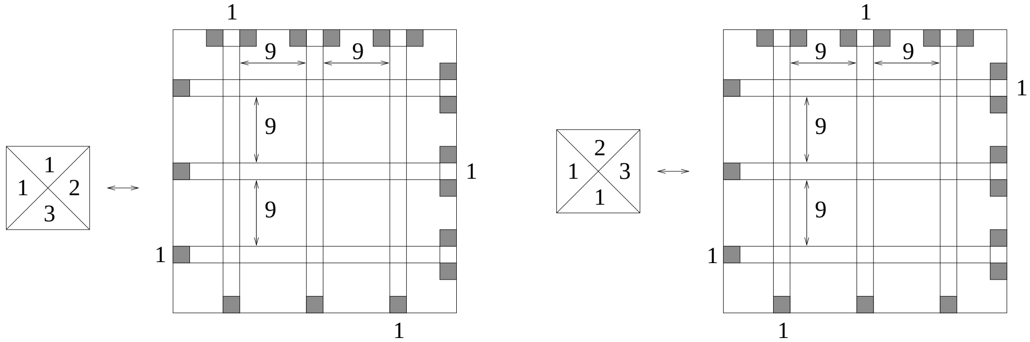
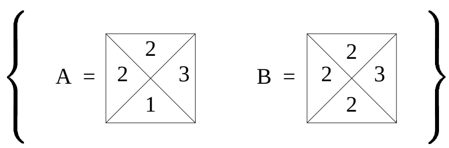


Figure 3. Tiles coded in the border conditions of B_0.

3. Circuit Coding

Given an arbitrary set of tiles $\mathcal{T}$, it can be coded in linear time (on $|\mathcal{T}|$) as a circuit lying on a $m \times m$ lattice square (m even). The idea is to represent each tile as a signal generator, to force the system to have only one of them in *on* state (the others *off*), and to join the generators (by conductors) with the corresponding border colors. Notice that, considering the square lattice as a chessboard (m even), we may assume the conductors arriving at black cells on the bottom and left borders.

Example. To the set of admissible tiles



we associate the following circuit

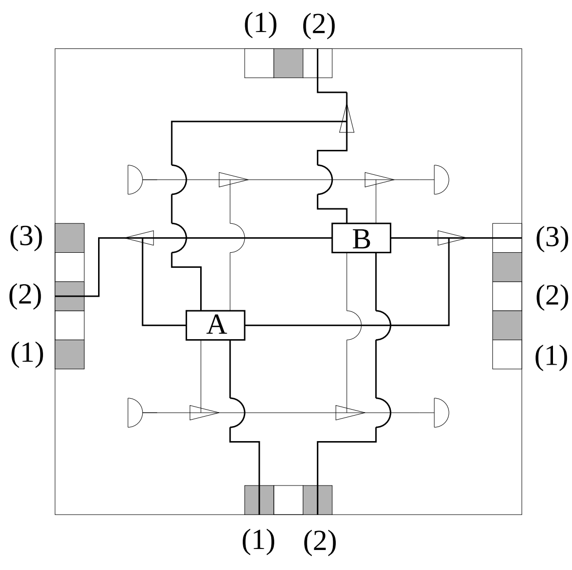


Figure 4. Circuit simulating the tile set $\{A, B\}$.

In this diagram we distinguish several units to be modeled in a $\mathcal{T}_0$-rotating board: *conductors, empty cells, filters* (two horizontal conductors with a left *origin* and a right *end* represented by semi-circles). Finally, two conductors may converge in *merge* units (arrow heads pointing in the sense of the signal conduction).

4. $\mathcal{T}_0$ - Rotating Subboard Assignment

First of all we refine the square in which the circuit has been drawn by transforming each cell into a 5×5 square $S_{5\times5}$. On the other hand, we associate colors black and white (as in a chessboard) to each cell of this new $(5m) \times (5m)$ square. In order to generate the $\mathcal{T}_0$-rotating board B_0 of size $(5m) \times (5m)$ we assign to each $S_{5\times5}$ a $\mathcal{T}_0$-rotating subboard. This assignment, which is explained in the following, depends on the circuit function of the cell represented by $S_{5\times5}$:

i. Empty cell: The assignment appears in Figure 5.

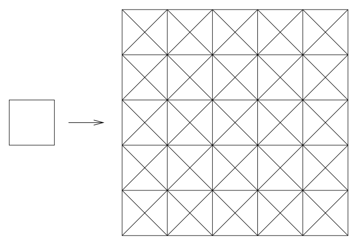


Figure 5. $\mathcal{T}_0$-rotating subboard assignment for an empty cell.

ii. Conductor: In order to simulate the circuit operation of signal conduction we consider as reference direction the one that goes from the generator to the borders, we assign to each circuit cell a $\mathcal{T}_0$-rotating subboard with only two possible tilings and we use the following key convention: a (conductor) subboard transmits signal 1 when the non null tiles located on white cells point their 1 value to the next cell. Otherwise, it transmits signal 0 (see Figures 6, 7 and 8).

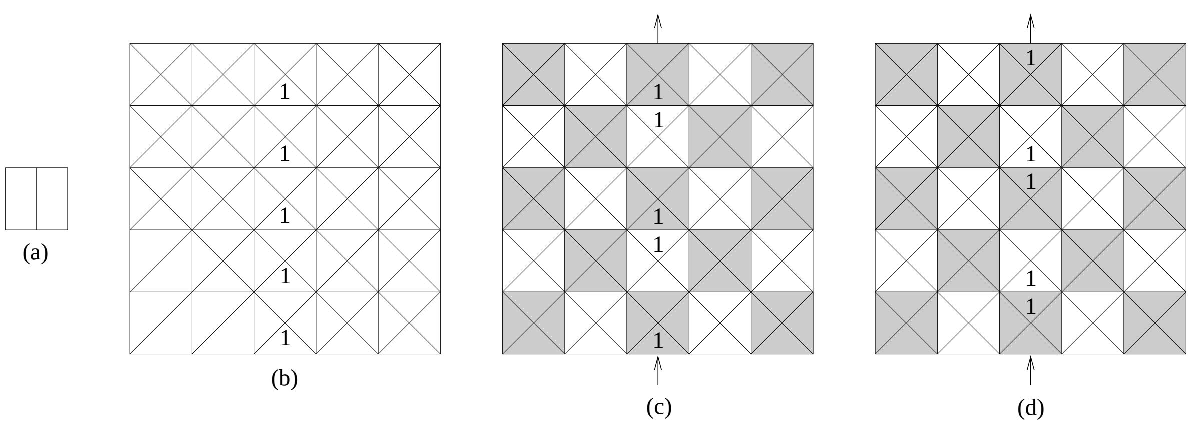


Figure 6. (a) Conductor cell. (b) Subboard assignment. (c) Signal 1 transmission. (d) Signal 0 transmission.

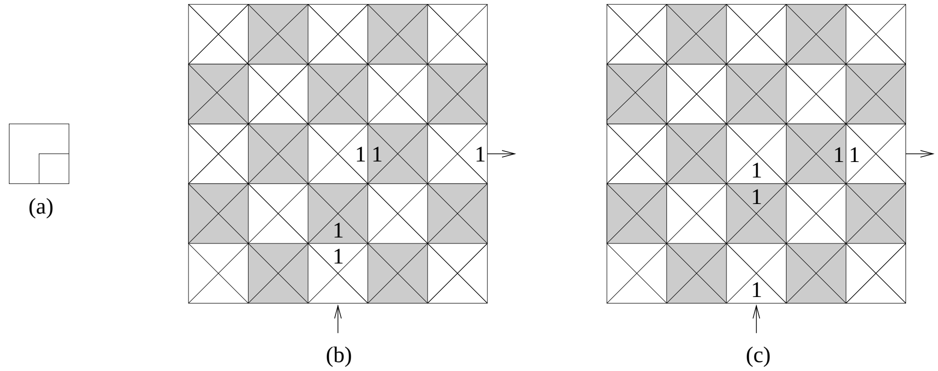


Figure 7. (a) Conductor cell. (b) Signal 1 transmission. (c) Signal 0 transmission.

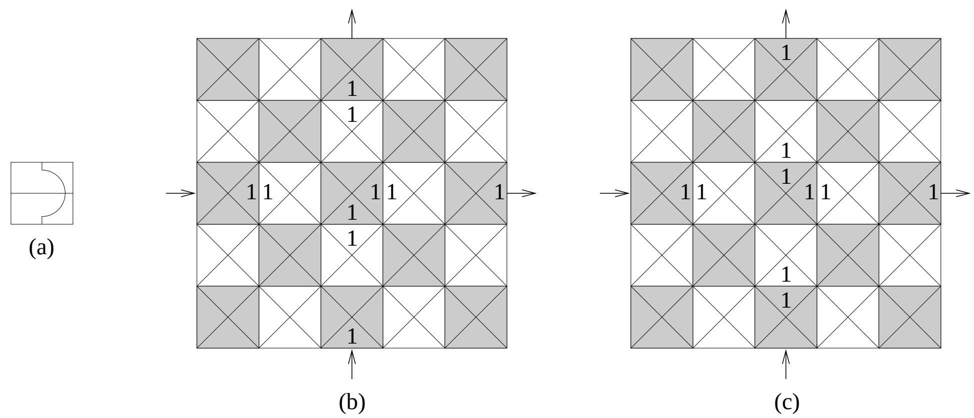


Figure 8 (a) Conductor crossing cell. (b) Signal 1 (vertically) and signal 0 (horizontally) transmission. (c) Signal 0 (vertically) and signal 0 (horizontally) transmission.

Notice that previous assignments are *independent* of the $S_{5\times5}$ alternation of colors. More precisely: no matter what the $S_{5\times5}$ center cell color is (black or white), the tilings will always correspond to one of the possible signal transmissions.

iii. Generator: The generator is simulated in two consecutive $S_{5\times5}$'s (see Figure 9). It produces six signals: four to establish the border colors and other two to inhibit other generators.

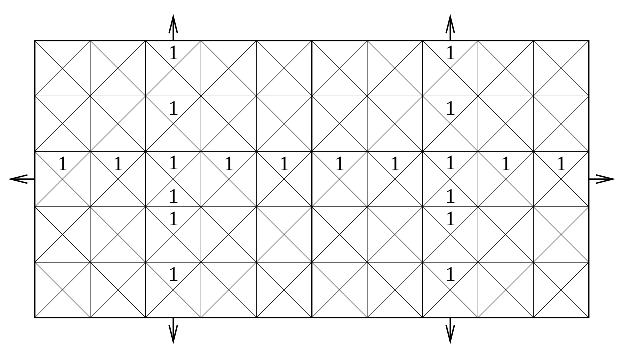


Figure 9. The two consecutive $\mathcal{T}_0$-rotating subboards that simulate a generator.

As in the conductors case, it doesn't matter what is the alternation of colors in which these subboards lie. More precisely, in order to obtain a tiling we have, in terms of signals, the only two possibilities of Figure 10.

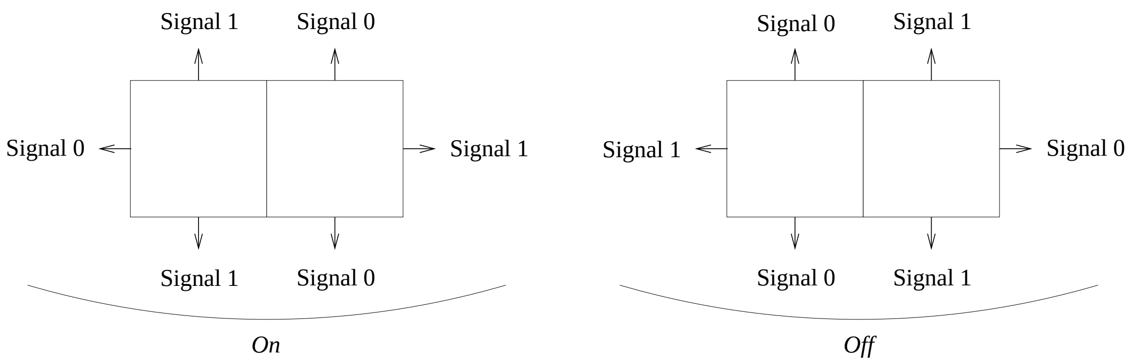


Figure 10.

Example. If the left center cell is black and the right one is white the *on* state (i.e. activation) is given in Figure 11.

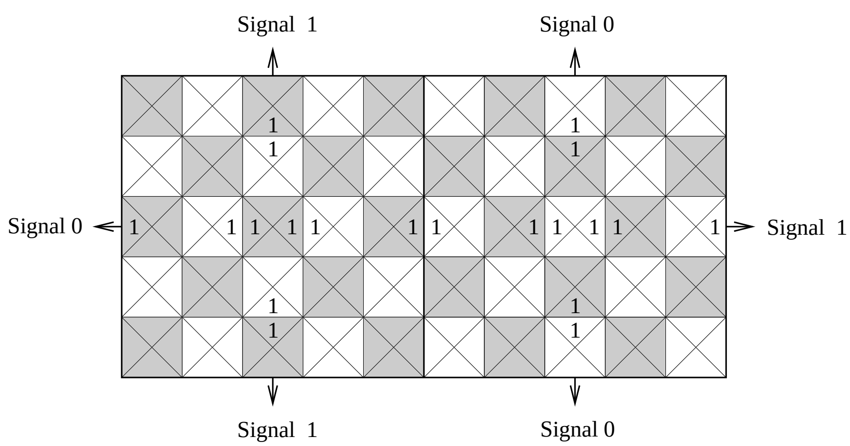


Figure 11. A generator in the *on* state.

iv. Filter: Both the up and bottom filters contain conductor cells (assignment already shown in (ii)), merge units (to be explained in (v)), an *origin* cell and an *end* cell.

The up-origin is forced to generate signal 1 and the up-end is forced to receive signal 0 (these units are not simple conductors because they must interact with just one neighbor cell). For both the up-origin and the up-end there are two $\mathcal{T}_0$-rotating subboards possible assignment depending if the center cell of the $S_{5\times5}$ in which the $\mathcal{T}_0$-rotating subboard lies is white or if it is black (see Figures 12 and 13). Notice that the assignment is *dependent* on the alternation of colors.

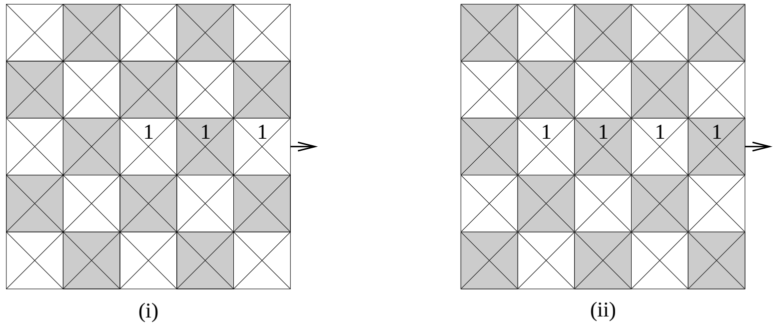


Figure 12. Up-origin. (i) White center cell. (ii) Black center cell.

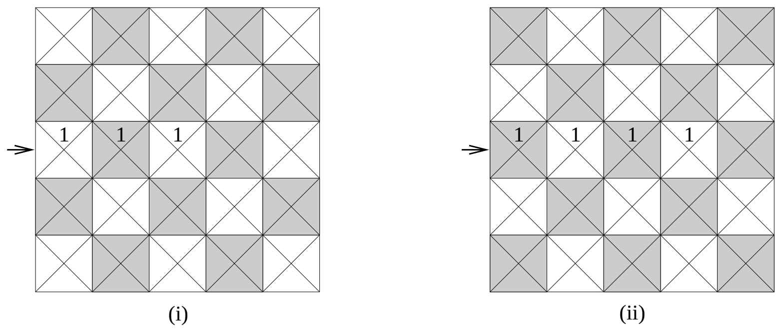


Figure 13. Up-end. (i) White center cell. (ii) Black center cell.

The bottom-origin generates signal 0 and the bottom-end receives signal 1. Their constructions are similar.

Comment. The analysis of two different cases depending on whether the center cell is black or white could be avoided by drawing the circuit in a more particular way (for instance by fixing the appearance of the up-origin in a black cell). This approach is equivalent: it simplifies the assignment but it complicates de circuit design.

v. Merge units: These are the elements which force the system to have only one generator in the *on* state. More precisely, they simulate one of the two logical gates:

a)	input signals	output signal	b)	input signals	output signal
	00	0		00	ϕ
	01	1		01	0
	10	1		10	0
	11	ϕ		11	1

where ϕ is non defined because this situation can not appear.

The assignment, as in the filter case, depends on the color alternation. In Figure 14 we describe the simulation of the logical gate (a):

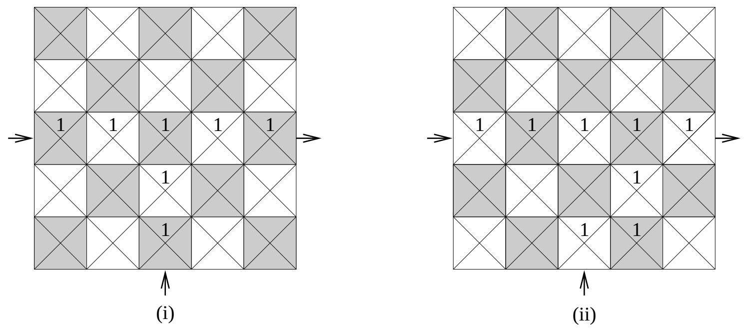


Figure 14. Merge unit of type a. (i) Black center cell. (ii) White center cell.

Clearly, for this simulation the input $\{1, 1\}$ does not give a tiling. The impossibility for the black center case appears in Figure 15:

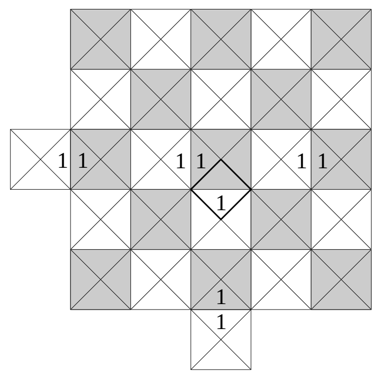


Figure 15. No tiling for the $\{1, 1\}$ input.

The case (b) is similar.

In our construction, in order to obtain a tiling, there must be a unique generator in *on* state. It is going to send a color 1 to each of the borders (corresponding to signals 1 sent to the right and up borders and to signals 0 sent to the left and bottom borders). Finally, by the fact that the reduction is linear, we conclude the theorem.

5. Undecidability and Complexity Results

We have already shown how to code a set of Wang tiles $\mathcal{T}$ by a $\mathcal{T}_0$-rotating board in linear time. Therefore, any result concerning the tilability of regions of the plane by $\mathcal{T}$ may be restated in the tile rotational context. More precisely,

Corollary. Let R be an arbitrary region and let $\mathcal{T}$ be an arbitrary finite set of tiles. We can build in linear time on $(|\mathcal{T}| + |R|)$ a rotating region $T_0 \in \mathcal{T}_0^{\widetilde{R}}$ (with $\widetilde{R}$ being a simple refinement of R) such that the existence of a tiling T_θ is equivalent to the tilability of R by $\mathcal{T}$.

Proof. It suffices to replace each cell of the region R by the $\mathcal{T}_0$-rotating board B_0 that codes $\mathcal{T}$. Therefore, this rotating region $T_0 \in \mathcal{T}_0^{\widetilde{R}}$ (where $|\widetilde{R}| = |B_0| \times |R|$) admits a tiling T_θ if and only if R is tilable by $\mathcal{T}$. □

Corollary. PERIODIC-ROT ($\mathcal{T}_0$) is undecidable and BOUND-ROT($\mathcal{T}_0$) is NP-complete.

Proof. By previous corollary and by the fact that the tiling problem is undecidable in its unbounded version [Be66] and NP-complete in its bounded one [Lew78]. □

Comment. Notice that the set $\mathcal{T}_0$ does not depend on $\mathcal{T}$. On the other hand, the number of colors of $\mathcal{T}_0$ is minimal (with one color we can code nothing).

Comment. A set of tiles is said to be *aperiodic* if it admits only non periodic tilings of the plane. In [Ber66,Rob71] it was proved that this kind of set effectively exists. Let us code a given aperodic set of tiles $\mathcal{T}$ by a $\mathcal{T}_0$-rotating board B_0. Let us now define the $\mathcal{T}_0$-rotating plane T_0 as the (periodic) repetition of B_0. It follows that the non-periodicity of the tilings of the plane by $\mathcal{T}$ is reflected in the non-periodicity of the bi-sequences θ's for which T_θ is a tiling. Notice finally that there is no periodic θ for which T_θ is a tiling.

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References.

[Be66] Berger R., *The undecidability of the domino problem*, Mem. Amer. Math. Soc. 66 (1966).

[vEB83] van Emde Boas P., *Dominoes are forever*, GTI Workshop, Paderborn, (1983), 75-95.

[GR97] Goles E., Rapaport I., *Complexity of tile rotation problems*, Theoretical Computer Science, 188, (1997), 129-159.

[EPH94] Etzion-Petruschka Y., Harel D., *On the solvability of domino snake problems*, Theoretical Computer Science, 131, (1994), 243-269

[Har86] Harel D., *Effective transformations on infinite trees with applications to high undecidability, dominoes and fairness*, J. ACM 33 (1986), 224-248.

[Lew78] Lewis H., *Complexity of solvable cases of the decision problem for predicate calculus*, Proc. 19th Symp. on Foundations of Computer Science (1978), 35-47.

[Mat91] Matamala M., *Complejidad de redes neuronales no lineales y aplicaciones a la optimización*, Engineering Thesis, Depto. Ingeniería Matemática, U. de Chile, (1991).

[Ro91] Robson M., *Le recouvrement d'une figure par h_n, v_m est NP-complet*, Beauquier D. (Ed.) Actes du Séminaire "Polyominoes et Pavages", Université de Créteil, (1991).

[Rob71] Robinson R., *Undecidability and nonperiodicity for tilings of the plane*, Inventiones Mathematicae 12, (1971), 177-209.

[Wan61] Wang, H., *Proving theorems by pattern recognition*, H. Bell System Tech. J., 40 (1961), 1-41.