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Complexity of tile rotation problems[1]

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Abstract

In this paper we introduce tile rotation problems. The instances (or initial configurations) are tile assignments on a $(n \times n)$ lattice board, and the question to be answered is the following: does there exist any configuration obtained from the initial one by tile rotations only whose cost is less than a given bound? (notice that a zero-cost configuration corresponds to a perfect tiling). We prove here the NP-completeness for both the zero-cost problem (for a particular set of 5 tiles) and the minimization problem (for a particular set of 2 tiles). Finally, by showing the polynomiality of some subproblems, we establish complexity border results.

1. Introduction

Wang tiles are unit-sized squares with colored (integer) edges. The *tiling problem* was introduced by Hao Wang in [12] and since then, many different versions of it have been studied (*bounded tiling* [8], *recurring tiling* [6], *domino snake problem* [4]).

Two *rotation problems* are introduced in this paper. For both of them the instances (or initial configurations) are tile assignments on a $(n \times n)$ lattice board, and their respective questions are the following:

(i) *Tile rotation problem* (TR): Does there exist any configuration obtained from the initial one by tile rotations only which corresponds to a perfect tiling?

(ii) *Minimization tile rotation problem* (MTR): Does there exist any configuration obtained from the initial one by tile rotations only whose cost is less (where the cost function is defined as the module of adjacencies tile differences) than a given bound?

In Section 3 we prove the TR NP-completeness for a particular set of 5 tiles. Notice that, in a sense, TR is a restricted version of the classical (and known NP-complete)

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bounded tiling problem [8]. We also introduce a TR generalization denoted by TILE-EXT and we establish the following complexity border result: TILE-EXT with 3 degrees of freedom is NP-complete while TILE-EXT with 2 degrees of freedom is polynomial.

The minimization problem MTR appears in the context of integrated circuit CAD and it turns to be polynomial for a particular $(n \times 1)$ lattice board [9]. In Section 4 we prove its NP-completeness for a particular set of 2 tiles. We establish a complexity border result by showing that MTR is trivial when the set of admissible tiles is a singleton. On the other hand, we prove that MTR becomes also polynomial when tile assignments are defined on fixed width rectangles or on acyclic regions instead of square boards.

2. Definitions

We denote by $\mathscr{T} \subseteq \mathbb{Z}^4$ the set of admissible tiles, and by S_n the $n \times n$ board of unit squares (or cells). For each tile $e=(e_0,e_1,e_2,e_3) \in \mathscr{T}$ we define its usual rotations of $0°, 90°, 180°$, and $270°$ (see Fig. 1) as follows:

$$e(\varphi) = (e_{(0+\varphi) \bmod 4}, e_{(1+\varphi) \bmod 4}, e_{(2+\varphi) \bmod 4}, e_{(3+\varphi) \bmod 4}) \quad \varphi = 0,1,2,3.$$

The initial tile configuration $T_0 \in \mathscr{T}^{n^2}$ associates, to each cell of S_n, a tile belonging to $\mathscr{T}$.

We also say that T_θ is a legal configuration if it is obtained from T_0 by a rotation vector $\theta = (\theta_{ij}) \in \{0,1,2,3\}^{n^2}$ (i.e. $(T_\theta)_{ij} = (T_0)_{ij}(\theta_{ij})$). Notice that $T_{\theta=0}$ is the initial tile configuration.

We consider two tiles e and e' as adjacents if and only if they have a common side, and we define their local cost function $c(e,e')$ as the module of the difference of the adjacent sides (see Fig. 2):

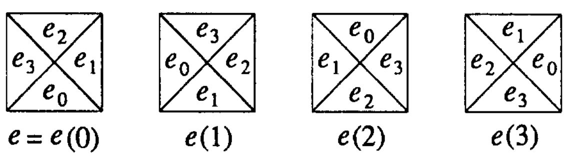


Fig. 1. Rotations of a tile.

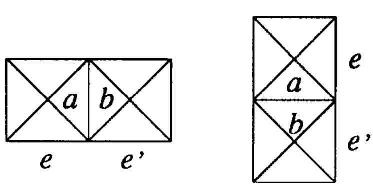


Fig. 2. $c(e,e') = |a-b|$.

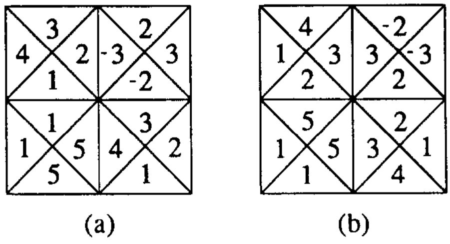


Fig. 3. (a) Initial configuration. (b) Legal configuration with cost function: $|5-3|+|3-3|+|2-5|+|2-2|=5$.

Fig. 4. Set of admissible tiles $\mathscr{T}_1$.

Given a configuration T_θ in S_n, the global cost function follows directly:

$$c(T_\theta)=\sum_{\substack{(i,j),(i',j')\\ \text{neighbors}}} c((T_\theta)_{ij},(T_\theta)_{i'j'}).$$

Example 1. In Fig. 3 appears an initial configuration defined on a (2×2) lattice board and a corresponding legal configuration:

Definition 1 (*Tile rotation problem*).

TR($\mathscr{T}$) *Instance*: $T_0 \in \mathscr{T}^{n^2}$.
Question: $\exists\theta\in\{0,1,2,3\}^{n^2}$ such that $c(T_\theta)=0$?

Definition 2 (*Minimization tile rotation problem*).

MTR($\mathscr{T}$) *Instance*: $k_1\in\mathbb{N}, T_0\in\mathscr{T}^{n^2}$.
Question: $\exists\theta\in\{0,1,2,3\}^{n^2}$ such that $c(T_\theta)\leqslant k_1$?

Comment 1. Notice that, for any particular set of admissible tiles $\mathscr{T}^*$,TR($\mathscr{T}^*$) is a subproblem of MTR ($\mathscr{T}^*$).

3. Complexity of the tile rotation problem

Let $\mathscr{T}_1$ be the set of tiles given in Fig. 4.

The goal of this section is to prove the next theorem.

Theorem 1. TR($\mathscr{T}_1$) *is* NP-*complete.*

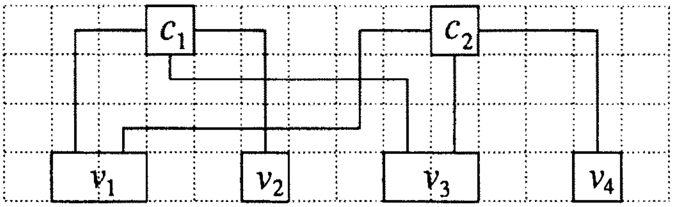


Fig. 5. Codification of a 1-in-3 SAT instance.

As a corollary we can conclude the weaker but more useful result that follows:

Corollary 1. TR *is* NP-*complete when the set of admissible tiles is part of the instance.*

The proof of Theorem 1 is based on Robson work related to the tilability of polygonal regions by elementary rectangles [11]. It consists to codify an NP-complete Boolean-logic problem known as 1-in-3 SAT (see [5]) as a circuit lying on the plane, and then to simulate the circuit by a tile assignment.

Considering a *clause* as a subset of Boolean variables and a *formula* as a collection of clauses, the 1-in-3 SAT definition is the next one:

Definition 3. 1-in-3 SAT.

1-*in*-3 *SAT* *Instance*: Set of Boolean variables V, formula C over V such that $\forall c \in C$, $|c| = 3$.
Question: $\exists$ any truth assignment on V such that each clause $c \in C$ contains exactly one true value?

The proof of Theorem 1 is shown in the following.

3.1. Codification of a 1-in-3 SAT instance

The key of the proof is to codify boolean formulas (which correspond to arbitrary instances of 1-in-3 SAT) as *electric circuits* drawn on a lattice board and constituted by signal *generators* (variables), signal *conductors*, and *acceptators* which accept as input the $\{1,0,0\}$ set of signals only (clauses).

Example 2. The Boolean formula $C = \{\{v_1, v_2, v_3\}, \{v_1, v_3, v_4\}\}$ can be codified as in Fig. 5.

Comment 2. Notice the existence of four types of cells in the board: *null, generator, conductor* and *acceptator*.

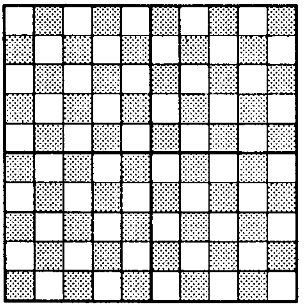


Fig. 6. Refinement of a 2 × 2 original board.

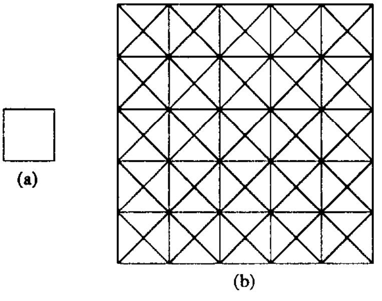


Fig. 7. (a) Null cell. (b) Sub-board tile assignment.

3.2. Reduction

Given an arbitrary 1-in-3 SAT instance codified as in Fig. 5, we must define a *board* and a *tile assignment* in order to build the TR instance.

3.2.1. Board

Refine the original board (in which the circuit is drawn) transforming each cell in a (5 × 5) sub-board. Associate colors black and white (as in a chessboard) to each generated sub-cell (Fig. 6).

Notice the existence of white-center and black-center sub-boards.

3.2.2. Tile assignment

The type of electric element codified in each sub-board (see Comment 2) determines its sub-cells tile assignment.

In order to simulate the electrical circuit operations of signal conduction (see Figs. 8–10), signal generation (see Fig. 11) and signal acceptation (see Fig. 12), we consider as reference direction the one that goes from the generator to the acceptator, and we use the following key convention:

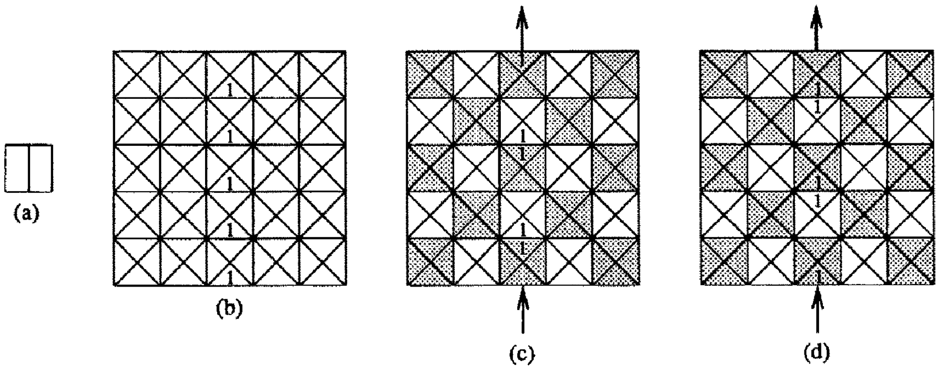


Fig. 8. (a) Conductor cell. (b) Sub-board tile assignment. (c) Signal 0 transmission. (d) Signal 1 transmission.

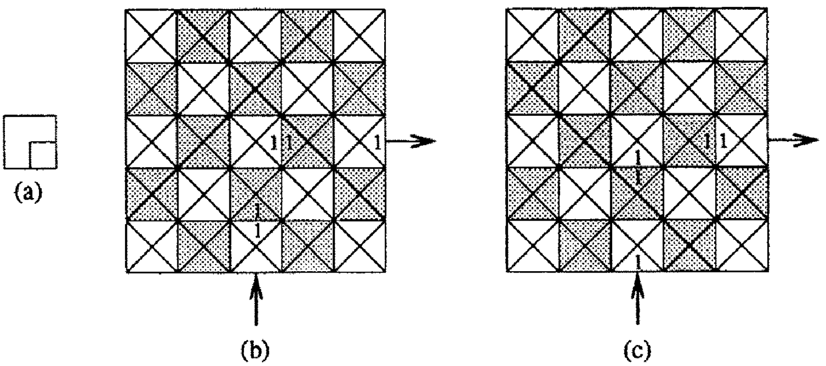


Fig. 9. (a) Conductor cell. (b) Signal 1 transmission. (c) Signal 0 transmission.

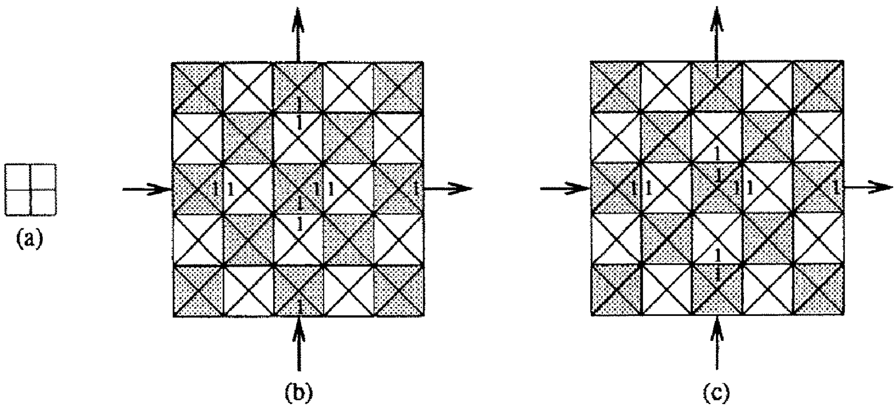


Fig. 10. (a) Conductor cell. (b) Signal 1 (vertically) and signal 0 (horizontally) transmissions. (c) Signal 0 (vertically) and signal 0 (horizontally) transmissions.

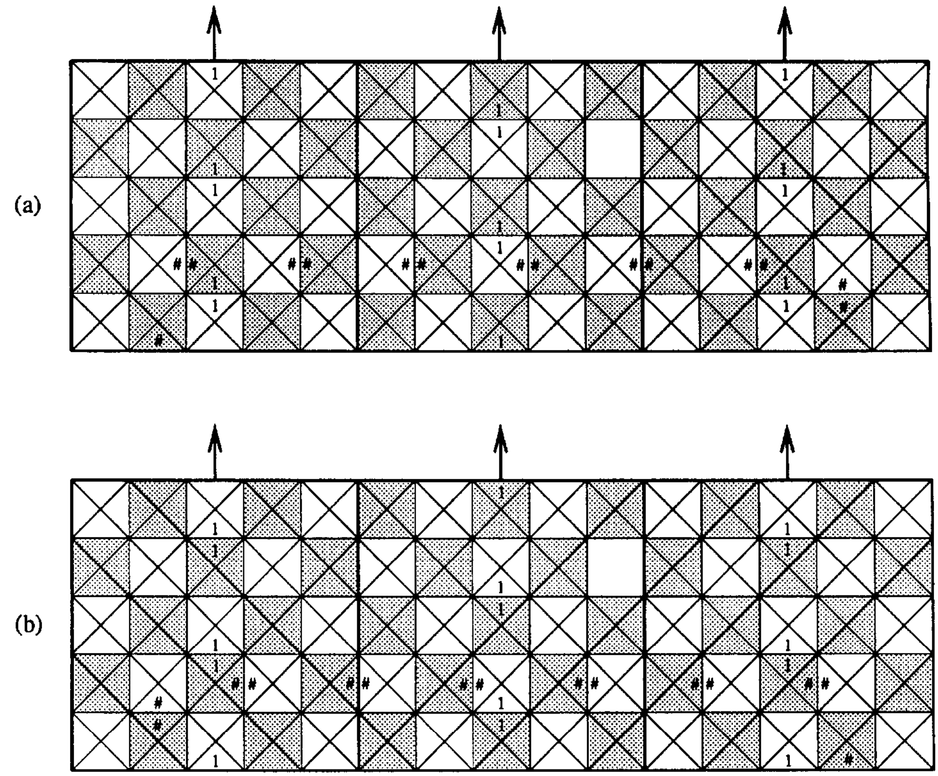


Fig. 11. Generator of size 3. (a) Signal 1 generation. (b) Signal 0 generation.

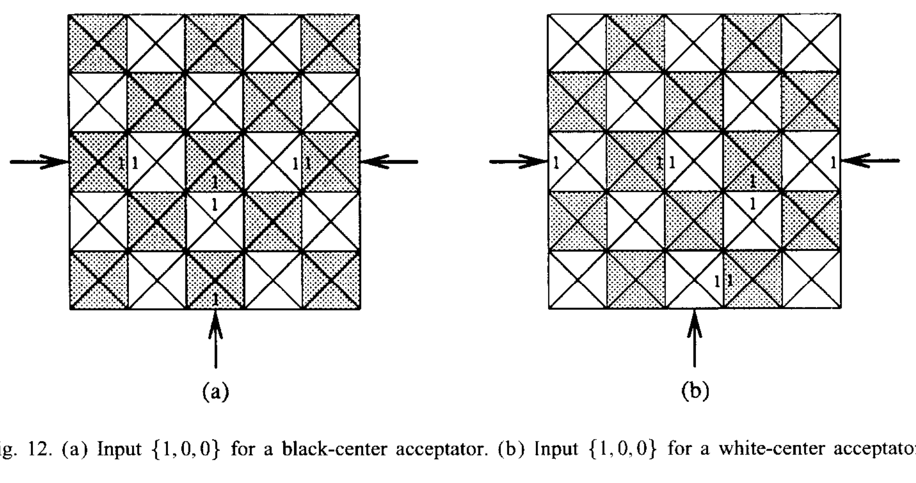


Fig. 12. (a) Input $\{1,0,0\}$ for a black-center acceptator. (b) Input $\{1,0,0\}$ for a white-center acceptator.

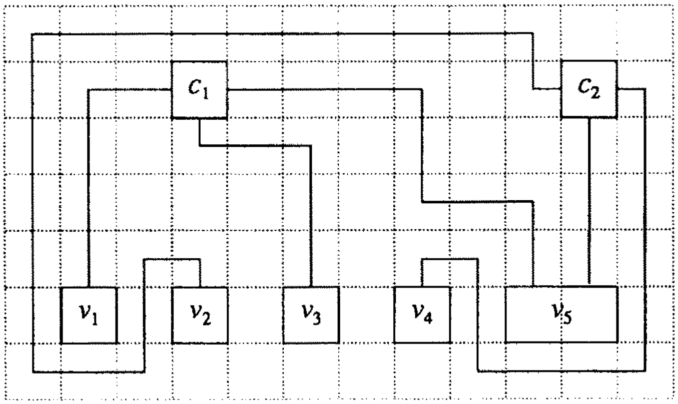


Fig. 13. Codification of the planar formula $C = \{\{v_1, v_3, v_5\}, \{v_2, v_4, v_5\}\}$.

A (conductor) sub-board transmits signal 1 *when the no null tiles located on white sub-cells point their* 1 *value to the next sub-cell. Otherwise, it transmits signal* 0 *(Fig.* 7).

(i) Null
(ii) Conductor
(iii) Generator
(iv) Acceptator

The effectiveness of the simulation, the NP-completeness of 1-in-3 SAT, and the fact that previous reduction is polynomially executable allow us to conclude Theorem 1. □

3.3. Degrees of freedom

Consider the TR generalization in which each cell has its own set of admissible tiles and rotations are not allowed. This new problem, denoted by TILE-EXT, is an extension of the classical tiling problem (where the set of admissible tiles is the same for every cell).

We say that TILE-EXT has $m \in \mathbb{N}$ degrees of freedom when the number of admissible tiles for each cell is at most m.

Lemma 1. *TILE-EXT with* 3 *degrees of freedom is* NP-*complete.*

Proof. Notice that by Theorem 1 we can only conclude the NP-completeness of TILE-EXT for 4 degrees of freedom. However, the planar version of 1-in-3 SAT (restricted version in which the formulas admit a planar codification as in Fig. 13) is still NP-complete (see [7]). That means that, without loss of generality, we can always consider electric circuits without conductor intersections. More precisely, we have removed the only source of 4 degrees of freedom.

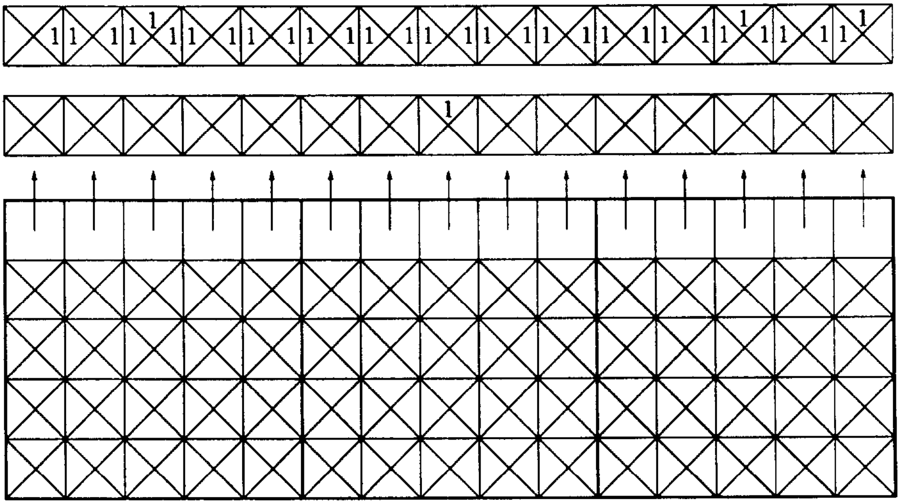


Fig. 14. Generator of size 3 for TILE-EXT (with at most two possible tiles for each cell).

The unique difference in this new type of circuit lies in the fact that, in order to have planarity, we cannot force the generators and the acceptators to be located on the bottom and on the top of the board, respectively.

Nevertheless, with respect to the acceptators we can maintain previous simulation because their up border was always the empty color.

In the case of the generators, even if they are not located on the bottom of the board, the TILE-EXT context allow us to simulate them with at most two possible tiles for each cell (as in Fig. 14) by having always the empty color on the bottom border. □

Theorem 2. *TILE-EXT with* 3 *degrees of freedom is* NP-*complete while TILE-EXT with* 2 *degrees of freedom is a polynomial problem.*

Proof. From Lemma 1 we have that TILE-EXT with 3 degrees of freedom is NP-complete. On the other hand, it is easy to reduce TILE-EXT with 2 degrees of freedom into 2-SAT (polynomial, see [3]) as in Example 3. □

Example 3. The tilability of the TILE-EXT instance (with 2 degrees of freedom) that appears in Fig. 15 is equivalent to the satisfiability of the following 2-SAT instance:

$$\begin{aligned} C = {} & (x_{11} \vee y_{11}) \wedge (x_{21} \vee y_{21}) \wedge (x_{12} \vee y_{12}) \wedge (x_{22} \vee y_{22}) \\ & \wedge (\bar{x}_{11} \vee \bar{y}_{21}) \wedge (\bar{y}_{11} \vee \bar{x}_{21}) \\ & \wedge (\bar{x}_{11} \vee \bar{x}_{12}) \wedge (\bar{y}_{11} \vee \bar{x}_{12}) \wedge (\bar{y}_{11} \vee \bar{y}_{12}) \\ & \wedge (\bar{x}_{21} \vee \bar{x}_{22}) \wedge (\bar{y}_{21} \vee \bar{x}_{22}) \wedge (\bar{y}_{21} \vee \bar{y}_{22}) \\ & \wedge (\bar{x}_{12} \vee \bar{y}_{22}) \wedge (\bar{y}_{12} \vee \bar{x}_{22}). \end{aligned}$$

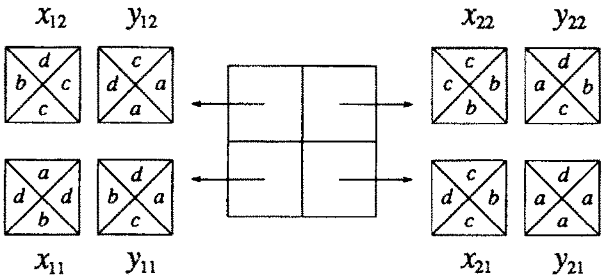


Fig. 15. TILE-EXT instance with 2 degrees of freedom.

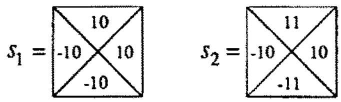


Fig. 16. Set of admissible tiles $\mathscr{T}_2 = \{s_1, s_2\}$.

4. Complexity of the minimization tile rotation problem

In this section we analyze the complexity of the minimization tile rotation problem (see Definition 2). More precisely, denoting as $\mathscr{T}_2$ the set of admissible tiles of Fig. 16, the goal is to prove next theorem.

Theorem 3. MTR($\mathscr{T}_2$) *is* NP-*complete.*

As a corollary we can conclude the weaker but more useful result that follows:

Corollary 2. *MTR is* NP-*complete when the set of admissible tiles is part of the instance.*

We prove Theorem 3 by reducing a physical problem known as spin glasses (SG) into MTR($\mathscr{T}_2$). In SG the instances are spin interactions (weighted arcs in a bipartite graph) and the question is the following: does there exist any spin orientation (assignments of $\pm$ values of the set of nodes) that maximizes some global energy? The spin interactions are, in fact, matrices. Therefore, it is natural to codify them in a two dimensional structure.

The idea of the proof is to represent the spin interactions as tiles in an initial configuration $\mathscr{T}_0$ that verifies:

- In order to minimize the cost function, we can restrict the process of searching θ to a *feasible set.*
- There is a one to one relation between the feasible θ's and the spin orientations.
- It is equivalent to maximize the energy of the spin glasses system and to minimize the cost function over the feasible configurations.

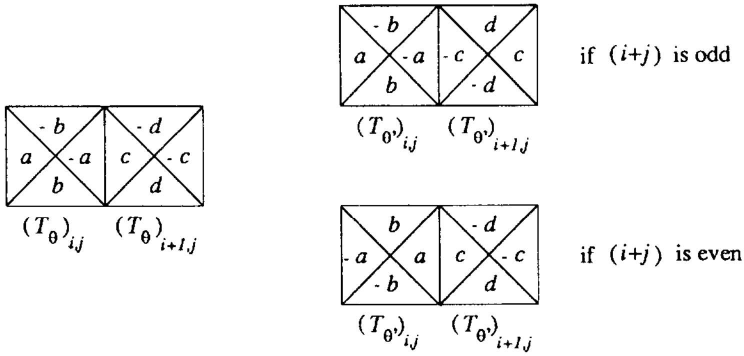


Fig. 17. Equivalence between c_- and c_+.

It is straightforward that MTR belongs to NP. In fact, if we choose *nondeterministically* a rotation vector θ, the time required to compute $c(T_\theta)$ and compare it with k_1 is polynomial in $(|n|^2 + |k_1|)$.

On the other hand, from now on, we will consider the cost function $c(e,e')$ as the module of the *sum* of the adjacent sides. This approach is justified by next lemma:

Lemma 2. *Let* $\mathscr{T} \subset \{(a_0,a_1,a_2,a_3) \in \mathbb{Z}^4\colon\ a_0 = -a_2 \wedge a_1 = -a_3\}$. *Let* T_0 *be an initial tile configuration. If we consider the costs functions* c_- *and* c_+ *defined with the difference and the sum respectively, it holds*:

$$[\exists \theta \text{ such that } c_-(T_\theta) \leqslant k_1] \Leftrightarrow [\exists \theta' \text{ such that } c_+(T_{\theta'}) \leqslant k_1].$$

Proof. It suffices to prove that the following rotations verify $c_-(T_\theta) = c_+(T_{\theta'})$:

$$\theta'_{ij} = \begin{cases} \theta_{ij} & \text{if } (i+j) \text{ is odd,} \\ (\theta_{ij}+2) \bmod 4 & \text{if } (i+j) \text{ is even.} \end{cases}$$

In fact, for tiles horizontally adjacents (see Fig. 17) we have

$$c_-((T_\theta)_{ij}, (T_\theta)_{i+1,j}) = |c - (-a)|$$

and

$$c_+((T_{\theta'})_{ij}, (T_{\theta'})_{i+1,j}) = |a + c|.$$

The argument is identical for tiles vertically adjacents. □

Finally, we will represent the positive components of a tile by *heads* of bold arrows and the negative ones by *tails* of bold arrows. By coding ten bold arrows by a single normal one, we have for the set of tiles $\mathscr{T}_2 = \{s_1, s_2\}$ the representation of Fig. 18.

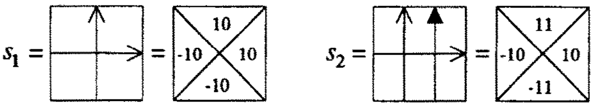


Fig. 18. Representation of $\mathscr{T}_2 = \{s_1, s_2\}$.

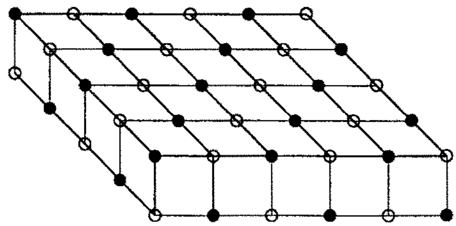


Fig. 19. Two-layers finite lattice.

4.1. *Auxiliar problem*

Considering that a finite bipartite-balanced graph $G=(X,Y,E)$ satisfies $E \subset X \times Y$ with $|X|=|Y|$, the formal definition of the already presented spin glasses problem is the following:

Definition 4 (*Spin glasses problem*).

SG Instance: $G=(X,Y,E)$ finite bipartite-balanced graph where degree$(i) \leqslant 5$ $\forall i \in X$, degree$(j) \leqslant 5$ $\forall j \in Y$; $(w_{ij})_{(ij)\in E} \in \{-1,1\}^{|E|}$ and $k_2 \in \mathbb{Z}$.

Question: $\exists x \in \{-1,1\}^{|X|}$, $y \in \{-1,1\}^{|Y|}$ such that

$$\mathscr{E}(x,y) = \sum_{(i,j)\in E} w_{ij} x_i y_i \geqslant k_2?$$

Previous problem appears in the framework of the physic model of spin glasses. In this context $\mathscr{E}$ corresponds to the energy associated to spin orientations (± 1 for each vertex in $X \cup Y$). It has been proved that this problem is NP-complete [1, 2]. In fact, the author proves that the problem is NP-complete for a particular bipartite-balanced graph consisting in a two-layers finite lattice (see Fig. 19).

4.2. *Polynomial transformation*

The polynomial transformation of an arbitrary SG-instance into a MTR-instance proposed here is the following (see Fig. 21):

- Size of the board $n = 3|X| + 1 = 3|Y| + 1$.
- Bound $k_1 = |E| + 2|X||Y| - k_2 = |E| + 2((n-1)/3)^2 - k_2$.
- In order to define the initial tile configuration T_0 we have to take in count the next two considerations:
  - Without loss of generality, we assume that the set of vertices $X \cup Y$ is given by $X = Y = \{1, \ldots, |X| = |Y| = ((n-1)/3)\}$.

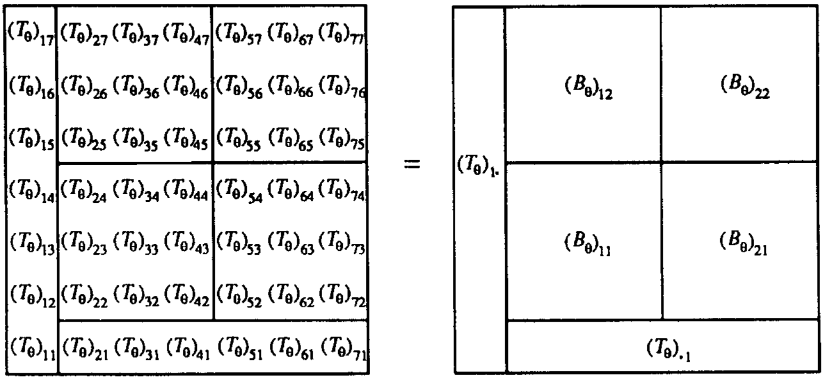


Fig. 20. Block partition for $n = 7$.

– We code a tile configuration T_0 in a suitable way by grouping the tiles in blocks $(B_\theta)_{(ij)\in\{1,\dots,(n-1)/3\}^2}$ as follows:

$$(B_\theta)_{ij} = \begin{bmatrix} (T_\theta)_{3i-1,3j+1} & (T_\theta)_{3i,3j+1} & (T_\theta)_{3i+1,3j+1} \\ (T_\theta)_{3i-1,3j} & (T_\theta)_{3i,3j} & (T_\theta)_{3i+1,3j} \\ (T_\theta)_{3i-1,3j-1} & (T_\theta)_{3i,3j-1} & (T_\theta)_{3i+1,3j-1} \end{bmatrix}.$$

Since the first row and column are outside the blocks, we also define:

$$(T_\theta)_{1\bullet} = ((T_\theta)_{1k})_{k=1,\dots,n},$$

$$(T_\theta)_{\bullet 1} = ((T_\theta)_{k1})_{k=2,\dots,n}.$$

So, we have the equivalence $T_\theta = ((T_\theta)_{1\bullet}, (T_\theta)_{\bullet 1}, B_\theta)$ (see Fig. 20).

Now, given the tile set $\mathscr{T}_2 = \{s_1, s_2\}$ and an arbitrary SG-instance, we build the initial tile configuration $T_0 = ((T_0)_{1\bullet}, (T_0)_{\bullet 1}, B_0)$ as follows (see Fig. 21):

(i) $((T_0)_{1k}) = ((T_0)_{1k}) = s_1$ for $k = 1, \dots, n$.

(ii) $((T_0)_{k1}) = ((T_0)_{k1}) = s_1$ for $k = 2, \dots, n$.

(iii) Three types of blocks $(B_0)_{ij}$ will be built (depending whether (i,j) belongs to E and on the value of w_{ij}):

(a) If $(i,j) \notin E$:

$$(B_0)_{ij} = \begin{bmatrix} s_1 & s_1 & s_1 \\ s_1 & s_2 & s_1 \\ s_1 & s_1 & s_1 \end{bmatrix} = \mathscr{B}_1.$$

(b) If $w_{ij} = 1$:

$$(B_0)_{ij} = \begin{bmatrix} s_1 & s_1 & s_1 \\ s_1 & s_2 & s_1 \\ s_1 & s_2 & s_1 \end{bmatrix} = \mathscr{B}_2.$$

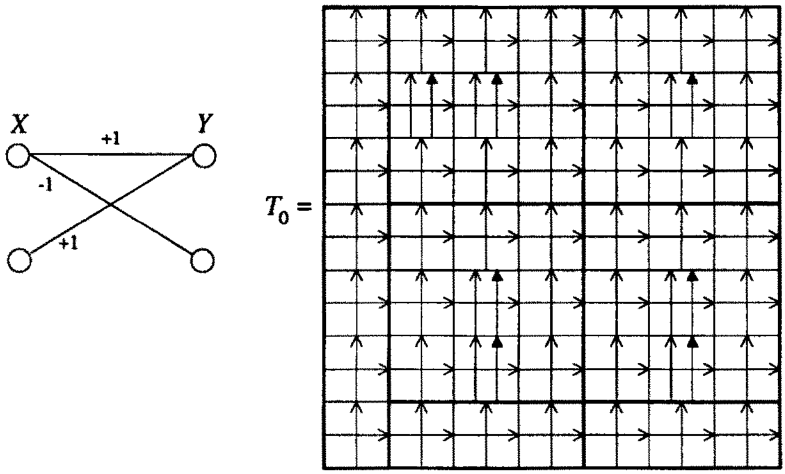


Fig. 21. Transformation of a finite bipartite-balanced graph into an initial tile configuration.

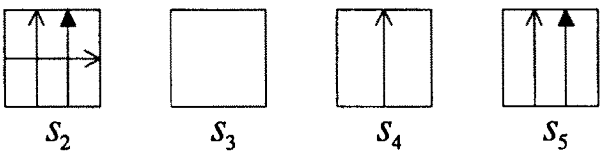


Fig. 22. Set of admissible tiles $\tilde{\mathcal{T}}_2$.

(c) If $w_{ij} = -1$:

$$(B_0)_{ij} = \begin{bmatrix} s_1 & s_1 & s_1 \\ s_2 & s_2 & s_1 \\ s_1 & s_2 & s_1 \end{bmatrix} = \mathcal{B}_3.$$

4.3. Auxiliar configurations

Let $\tilde{\mathcal{T}}_2 = \{s_2, s_3, s_4, s_5\}$ be the new set of admissible tiles that appears in Fig. 22:

Consider the following one-to-one relation between a $\mathcal{T}_2$-initial configuration T_0 and a $\tilde{\mathcal{T}}_2$-initial configuration T_0 (see Fig. 23):

(i) $(\tilde{B}_0)_{ij} = \tilde{\mathcal{B}}_k \Leftrightarrow (B_0)_{ij} = \mathcal{B}_k$ for $k = 1, 2, 3$, where

$$\tilde{\mathcal{B}}_1 = \begin{bmatrix} s_3 & s_4 & s_3 \\ s_4 & s_2 & s_4 \\ s_3 & s_4 & s_3 \end{bmatrix}, \quad \tilde{\mathcal{B}}_2 = \begin{bmatrix} s_3 & s_4 & s_3 \\ s_4 & s_2 & s_4 \\ s_3 & s_5 & s_3 \end{bmatrix}, \quad \tilde{\mathcal{B}}_3 = \begin{bmatrix} s_3 & s_4 & s_3 \\ s_5 & s_2 & s_4 \\ s_3 & s_4 & s_3 \end{bmatrix}.$$

Remark that, in terms of unit tiles, previous transformation is equivalent to replace all the s_1 tiles: the ones located at the corners of a block by s_3 and the others

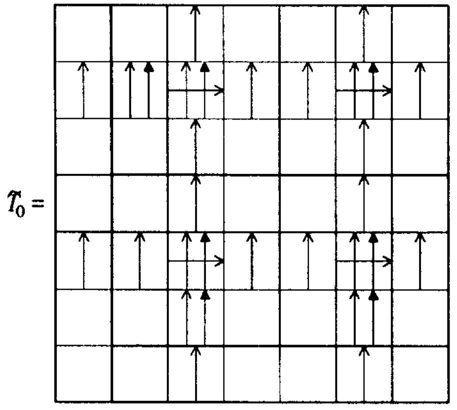


Fig. 23. $\tilde{\mathscr{T}}_2$-initial configuration $\tilde{T}_0$ associated with the $\mathscr{T}_2$-initial configuration of Fig. 21.

by s_4. With respect to the s_2 tiles, the ones not located at the center of a block are replaced by s_5.

(ii) $(\tilde{T}_0)_{1k} = \begin{cases} s_4 & \text{if } k = 0 \pmod 3, \\ s_3 & \text{otherwise,} \end{cases}$

$(\tilde{T}_0)_{k1} = \begin{cases} s_4 & \text{if } k = 0 \pmod 3, \\ s_3 & \text{otherwise.} \end{cases}$

4.4. Aligned configurations

Definition 5. Let $\tilde{T}_0$ be a $\tilde{\mathscr{T}}_2$-configuration. We say that $\tilde{T}_\theta$ is an *aligned configuration* if and only if there is no arrow in $\tilde{T}_\theta$ pointing to a null tile s_3. (Notice that $\tilde{T}_0$ is not an aligned configuration.)

Lemma 3. *Given a $\mathscr{T}_2$-configuration T_{θ_1}, there exists a rotation θ_2 such that $\tilde{T}_{\theta_2}$ is an aligned configuration and $c(\tilde{T}_{\theta_2}) \leqslant c(T_{\theta_1})$.*

Proof. We apply to an arbitrary $\mathscr{T}_2$-configuration T_{θ_1} the following procedure (see Fig. 24):

- *Step* 1: Erase every single arrow ($\uparrow$) which does not lie on a $3i$-column or a $3j$-row for $i,j \in \{1,\dots,(n-1)/3\}$.
- *Step* 2: For each tile (that after step 1 remains) s_2 which is not located at the center of a block, erase its single arrow ($\uparrow$) and then rotate it in one unit.

From previous procedure we see that:

(i) The output is an aligned configuration denoted by $\tilde{T}_{\theta_2}$.

(ii) With first step the global cost function increases its value *at most* 20 times the number of double arrows of T_{θ_1} which do not lie on a $3i$-column or a $3j$-row (for $i,j, \in \{1,\dots,(n-1)/3\}$). In step 2, however, the cost function decreases *at least* in the same quantity. So, $c(\tilde{T}_{\theta_2}) \leqslant (T_{\theta_1})$. $\square$

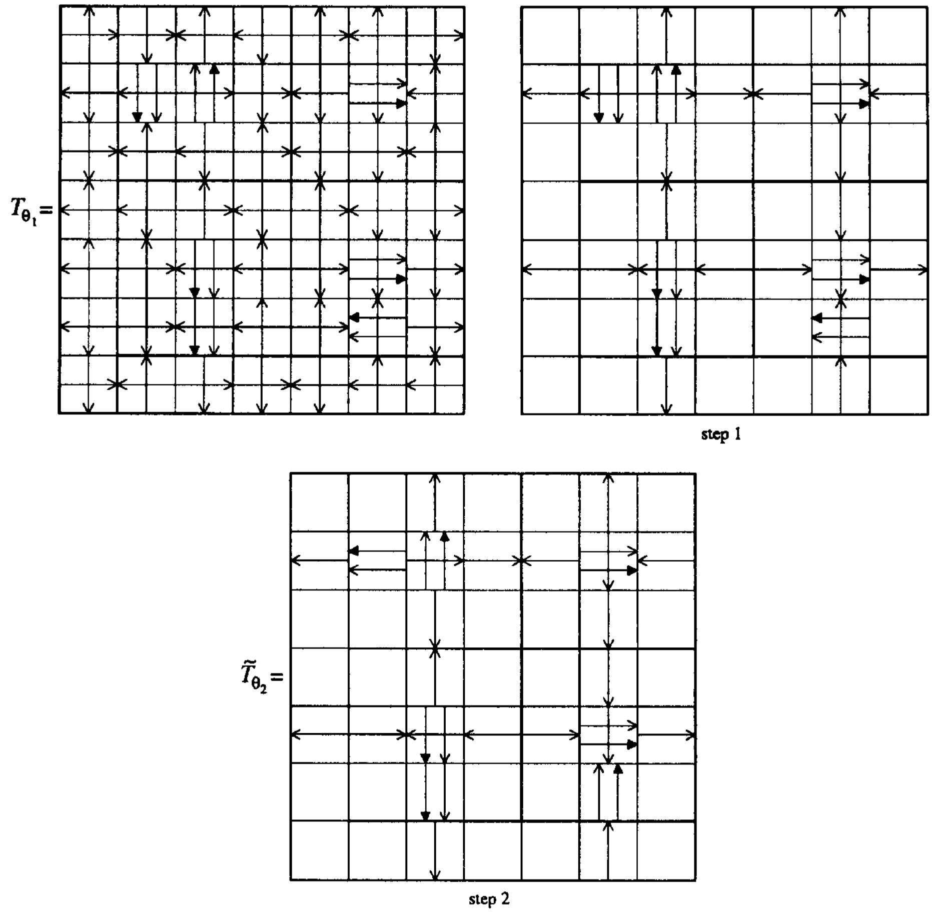


Fig. 24. Transformation of T_{θ_1} into an aligned configuration $\tilde{T}_{\theta_2}$.

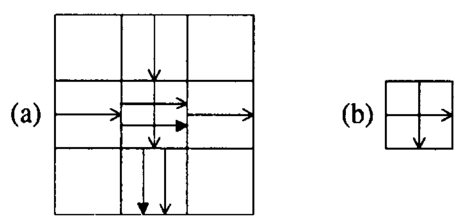


Fig. 25. (a) Feasible block. (b) Representation of a feasible block.

4.5. Semi-feasible configurations

Definition 6. Let $(\tilde{B}_\theta)_{ij}$ be a block of a $\tilde{\mathscr{T}}_2$-configuration. We say that it is a *feasible block* if and only if every two non-null adjacent tiles of $(\tilde{B}_\theta)_{ij}$ have colinear arrows pointing to the same direction (see Fig. 25(a)).

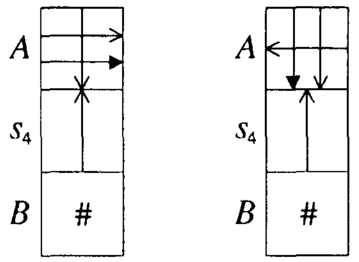


Fig. 26. Generic situation for a s_4 tile before being rotated.

Comment 3. Notice that, for a feasible block, the orientation of its tiles is fixed by the orientation of the central one. So, one may represent a feasible block as in Fig. 25(b):

Definition 7. Let $\tilde{T}_\theta$ be a $\tilde{\mathscr{T}}_2$-configuration. We say that $\tilde{T}_\theta$ is a *semi-feasible* configuration if and only if:

(i) $\tilde{T}_\theta$ is an aligned configuration.

(ii) $(\tilde{B}_\theta)_{ij}$ is a feasible block $\forall (i,j) \in X \times Y$.

Lemma 4. *Given an aligned configuration $\tilde{T}_{\theta_1}$, there exists a rotation θ_2 such that $\tilde{T}_{\theta_2}$ is a semi-feasible configuration and $c(\tilde{T}_{\theta_2}) \leqslant c(\tilde{T}_{\theta_1})$.*

Proof. The following procedure transforms an aligned configuration $\tilde{T}_{\theta_1}$ into a semi-feasible one $\tilde{T}_{\theta_2}$ without increasing the global cost function:

- Rotate in two units every s_4 and s_5 tile pointing in the opposite direction of its adjacent central block tile.

  With respect to previous procedure we see that:

  (i) The output is a semi-feasible configuration that we denote $\tilde{T}_{\theta_2}$.

  (ii) For the s_4 tiles to which the transformation is applied, we have the generic situation of Fig. 26 (where # represents an arbitrary tile).
  When we rotate in two units s_4, the cost function could not increase (due to its interaction with tile B) more than it decreases (due to its interaction with tile A) (Fig. 27).

  (iii) For the s_5 tiles to which the transformation is applied, we have the generic situation of Fig. 28 (where $|$ represents an arrow pointing up or down).
  When we rotate in two units s_5 (Fig. 29), the cost function could not increase (due to its interaction with tile B) more than it decreases (due to its interaction with tile A). □

It is shown in Fig. 30 the way previous procedure works when it is applied to an arbitrary aligned configuration $\tilde{T}_{\theta_1}$:

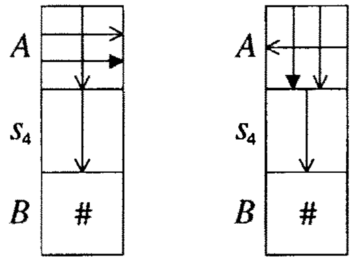


Fig. 27. Generic situation for a s_4 tile after being rotated.

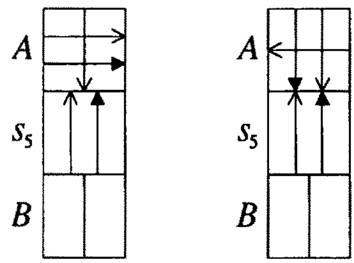


Fig. 28. Generic situation for a s_5 tile before being rotated.

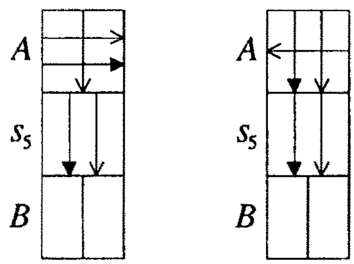


Fig. 29. Generic situation for a s_5 tile after being rotated.

4.6. Feasible configurations

Given a $\tilde{\mathscr{T}}_2$-configuration $\tilde{T}_\theta = ((\tilde{T}_\theta)_{1\bullet}, (\tilde{T}_\theta)_{\bullet 1}, \tilde{B}_\theta)$, we will separate the global cost function c into the following quantities:

$c_{\text{int}}((\tilde{B}_\theta)_{ij})$ is the total cost inside the block $(\tilde{B}_\theta)_{ij}$.
$c_{\text{int}}((\tilde{T}_\theta)_{1\bullet})$ is the total cost inside the first column.
$c_{\text{int}}((\tilde{T}_\theta)_{\bullet 1})$ is the total cost inside the first row.
$c_{\text{ext,h}}((\tilde{B}_\theta)_{ij})$ is the total cost on the bottom-horizontal border of the block $(\tilde{B}_\theta)_{ij}$.
$c_{\text{ext,v}}((\tilde{B}_\theta)_{ij})$ is the total cost on the left-vertical border of the block $(\tilde{B}_\theta)_{ij}$.

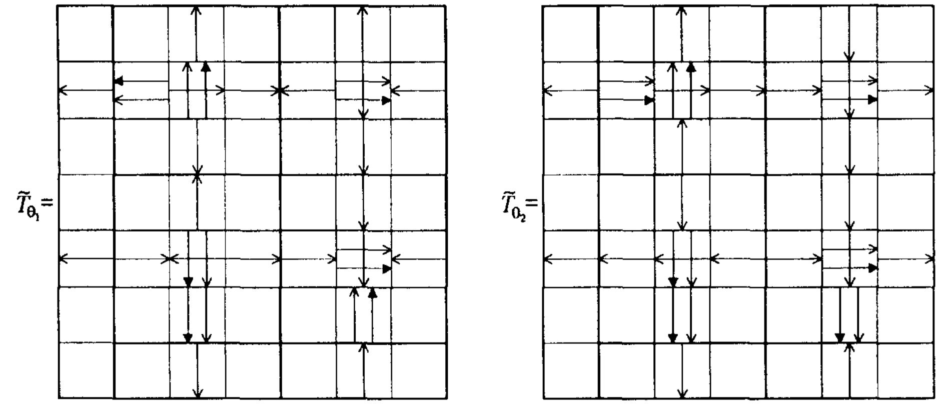


Fig. 30. Transformation of an aligned configuration into a semi-feasible one.

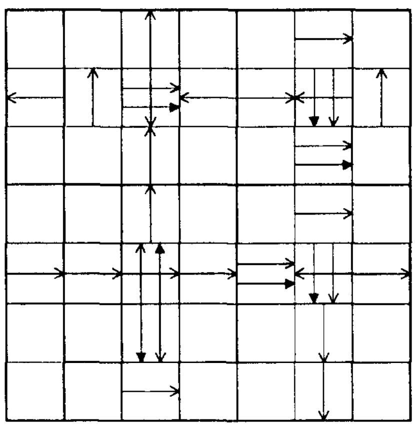


Fig. 31. Local evaluations of the cost function.

Finally, for a $\tilde{\mathscr{T}}_2$-configuration $\tilde{T}_\theta$, we define the global external cost in the kth row and the kth column of blocks ($k=1,\ldots,|X|=|Y|$) as follows:

$$c_k(\tilde{T}_\theta)=\sum_{i\in X} c_{\text{ext,v}}((\tilde{B}_\theta)_{ik}) + \sum_{j\in Y} c_{\text{ext,h}}((\tilde{B}_\theta)_{kj}).$$

In Fig. 31 appears a $\tilde{\mathscr{T}}_2$-configuration which satisfies:

$$c_{\text{int}}((\tilde{B}_\theta)_{11})=23, \quad c_{\text{int}}((\tilde{T}_\theta)_{1\bullet})=0, \quad c_{\text{int}}((\tilde{T}_\theta)_{\bullet 1})=20, \quad c_{\text{ext,h}}((\tilde{B}_\theta)_{22})=0,$$

$$c_{\text{ext,v}}((\tilde{B}_\theta)_{21})=1, \quad c_1(\tilde{T}_\theta)=12, \quad c_2(\tilde{T}_\theta)=30.$$

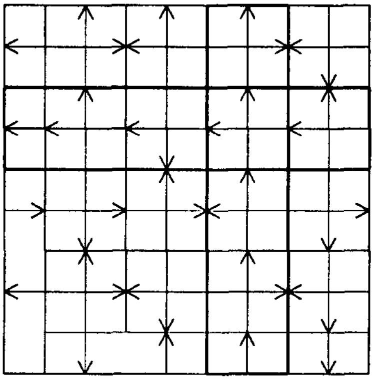


Fig. 32. 3-feasible configuration.

4.6.1. k-feasible configurations

Definition 8. Let $\tilde{T}_\theta$ be a semi-feasible configuration. We say that $\tilde{T}_\theta$ is a *k-feasible configuration* if and only if (see Fig. 32):

(i) Colinear arrows of the kth column of blocks and the arrow of tile $(\tilde{T}_\theta)_{3k,1}$ point to the same direction.

(ii) Colinear arrows of the kth row of blocks and the arrow of tile $(\tilde{T}_\theta)_{1,3k}$ point to the same direction.

For each $k \in \{1,\dots,|X|=|Y|\}$, we define the following sets of indexes:

$$C(-,k)=\{i\in X\colon\ w_{ik}=-1\},$$

$$C(+,k)=\{j\in Y\colon\ w_{kj}=+1\}.$$

Lemma 5. *For a semi-feasible configuration $\tilde{T}_\theta$, it holds:*

$$\begin{aligned} c_k(\tilde{T}_\theta) = {} & |C(+,k)| + |C(-,k)| + \sum_{i\in C(-,k)} |c_{\text{ext,v}}((\tilde{B}_\theta)_{ik}) - 1| \\ & + \sum_{j\in C(+,k)} |c_{\text{ext,h}}((\tilde{B}_\theta)_{kj}) - 1| \\ & + \sum_{i\in(X\setminus C(-,k))} c_{\text{ext,v}}(\tilde{B}_\theta)_{ik}) \\ & + \sum_{j\in(Y\setminus C(+,k))} c_{\text{ext,h}}((\tilde{B}_\theta)_{kj}). \end{aligned}$$

Proof. It suffices to note that:

$$c_{\text{ext,v}}((\tilde{B}_\theta)_{ik}) \geqslant 1 \quad \forall i\in C(-,k),$$

$$c_{\text{ext,h}}((\tilde{B}_\theta)_{kj}) \geqslant 1 \quad \forall j\in C(+,k). \qquad \square$$

Lemma 6. *For a semi-feasible configuration $\tilde{T}_\theta$, it holds:*

(i) $c_k(\tilde{T}_\theta) = |C(-,k)| + |C(+,k)| \Leftrightarrow \tilde{T}_\theta$ *is k-feasible.*

(ii) $c_k(\tilde{T}_\theta) \geqslant |C(-,k)| + |C(+,k)| + 20 \Leftrightarrow \tilde{T}_\theta$ *is not k-feasible.*

Proof. (i) Notice that a semi-feasible configuration $\tilde{T}_\theta$ is k-feasible if and only if $\forall (i,j) \in X \times Y$:

$$c_{\text{ext,v}}((\tilde{B}_\theta)_{ik}) = \begin{cases} 1 & \text{if } w_{ik} = -1, \\ 0 & \text{otherwise,} \end{cases}$$

$$c_{\text{ext,h}}((\tilde{B}_\theta)_{kj}) = \begin{cases} 1 & \text{if } w_{kj} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

From Lemma 5 we conclude that this is true if and only if:

$$c_k(\tilde{T}_\theta) = |C(-,k)| + |C(+,k)|.$$

(ii) ($\Rightarrow$) Direct consequence of part (i).

($\Leftarrow$) Since $\tilde{T}_\theta$ is not k-feasible, there exists at least one pair of colinear arrows in the kth column or row of blocks pointing in opposite directions. That is to say:

$$(\exists i \in X) \quad \{(w_{ik} = 1 \vee (i,k) \notin E) \wedge (c_{\text{ext,V}}((\tilde{B}_\theta)_{ik}) = 20)\}$$
$$\vee \{(w_{ik} = -1) \wedge (c_{\text{ext,V}}((\tilde{B}_\theta)_{ik}) = 21)\},$$

$$\vee(\exists j \in Y) \quad \{(w_{kj} = -1 \vee (k,j) \notin E) \wedge (c_{\text{ext,H}}((\tilde{B}_\theta)_{kj}) = 20)\}$$
$$\vee \{(w_{kj} = -1) \wedge (c_{\text{ext,H}}((\tilde{B}_\theta)_{kj}) = 21)\}.$$

If together with this we consider Lemma 5, we conclude:

$$c_k(\tilde{T}_\theta) \geqslant |C(+,k)| + |C(-,k)| + 20. \qquad \square$$

4.6.2. Frustrated blocks

Definition 9. Let $\tilde{T}_\theta = ((\tilde{T}_\theta)_{1\bullet}, (\tilde{T}_\theta)_{\bullet 1}, \tilde{B}_\theta)$ be a semi-feasible configuration. We say that $(\tilde{B}_\theta)_{ij}$ is a *frustrated block* (see Fig. 33) if and only if the double arrows of the central tile are not colinear with the double arrows of its adjacent tile (which codes an arc of the graph $G = (X, Y, E)$). In other words, $(\tilde{B}_\theta)_{ij}$ satisfies one of the following conditions:

$(\tilde{B}_0)_{ij} = \tilde{\mathscr{B}}_2$ and $\theta_{3i,3j}$ is an odd rotation $\vee$
$(\tilde{B}_0)_{ij} = \tilde{\mathscr{B}}_3$ and $\theta_{3i,3j}$ is an even rotation.

Obviously, a block of the form $\tilde{\mathscr{B}}_1$ is never frustrated.

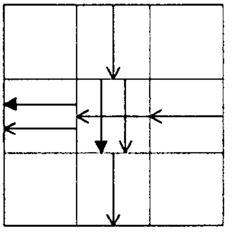


Fig. 33. Frustrated block: $(\tilde{\mathscr{B}}_\theta)_{ij} = \tilde{\mathscr{B}}_3$ and $\theta_{3i,3j} = 2$.

It follows from previous definition:

Lemma 7. *For a semi-feasible configuration $\tilde{T}_\theta$, it holds:*

$$c_{\text{int}}((\tilde{B}_\theta)_{ij}) = \begin{cases} 2 & \text{if } (i,j) \notin E, \\ 3 & \text{if } (i,j) \in E \text{ and } (\tilde{B}_\theta)_{ij} \text{ is frustrated}, \\ 1 & \text{if } (i,j) \in E \text{ and } (\tilde{B}_\theta)_{ij} \text{ is not frustrated.} \end{cases}$$

And: $c_{\text{int}}((\tilde{T}_\theta)_{1\bullet}) = c_{\text{int}}((\tilde{T}_\theta)_{\bullet 1}) = 0$.

4.6.3. Feasibility

Definition 10. Let $\tilde{T}_\theta$ be a $\tilde{\mathscr{T}}_2$-configuration. We say that $\tilde{T}_\theta$ is a *feasible configuration* if and only if every two non-null adjacent tiles of $\tilde{T}_\theta$ have colinear arrows pointing to the same direction. (Notice that if a tile configuration is feasible, then it is semi-feasible.)

The relation between feasible configurations and k-feasible configurations follows from the definition:

Lemma 8. *$\tilde{T}_\theta$ is feasible if and only if it is k-feasible* $\forall k \in \{1, \ldots, (n-1)/3\}$.

Comment 4. In order to build a feasible configuration $\tilde{T}_\theta$, it suffices to fix the orientation of the blocks in the diagonal of such configuration (see Fig. 34). It follows that the set of feasible configuration has cardinality $4^{|X|} = 4^{|Y|} = 4^{(n-1)/3}$.

Lemma 9. *For a feasible configuration $\tilde{T}_\theta$, it holds:*

$$c(\tilde{T}_\theta) = 2\left[\left(\frac{n-1}{3}\right)^2 + \sum_{\substack{w_{ij}=1 \\ \theta_{3i,3j}\ \text{odd}}} |w_{ij}| + \sum_{\substack{w_{ij}=-1 \\ \theta_{3i,3j}\ \text{even}}} |w_{ij}|\right].$$

Proof. For a $\tilde{\mathscr{T}}_2$-configuration, one may decompose the global cost as follows:

$$c(\tilde{T}_\theta) = \sum_{k \in \{1,\ldots,\frac{n-1}{3}\}} c_k(\tilde{T}_\theta) + \sum_{(i,j) \in X \times Y} c_{\text{int}}((\tilde{\mathscr{B}}_\theta)_{ij}) + c_{\text{int}}((\tilde{T}_\theta)_{1\bullet}) + c_{\text{int}}((\tilde{T}_\theta)_{\bullet 1})$$

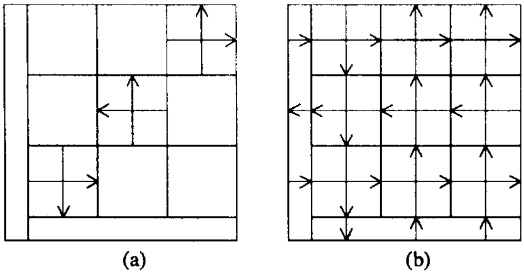


Fig. 34. Feasible configurations. (a) Orientation of the diagonal blocks. (b) Orientation of the whole configuration.

since $\tilde{T}_\theta$ is feasible, we have from Lemma 6:

$$\sum_{k\in\{1,\dots,\frac{n-1}{3}\}} c_k(\tilde{T}_\theta) = \sum_{k\in\{1,\dots,\frac{n-1}{3}\}} \{|C(+,k)| + |C(-,k)|\} = |E|$$

obtaining:

$$c(\tilde{T}_\theta) = |E| + \sum_{(i,j)\in X\times Y} c_{\text{int}}((\tilde{\mathscr{B}}_\theta)_{ij}) + c_{\text{int}}((\tilde{T}_\theta)_{1\bullet}) + c_{\text{int}}((\tilde{T}_\theta)_{\bullet 1})$$

considering now Lemma 7:

$$\begin{aligned} c(\tilde{T}_\theta) = {} & |E| + 2\,|\{(i,j)\in X\times Y\colon (i,j)\notin E\}| \\ & + 3\,|\{(i,j)\in E\colon (\tilde{\mathscr{B}}_\theta)_{ij} \text{ is frustrated}\}| \\ & + 1\,|\{(i,j)\in E\colon (\tilde{\mathscr{B}}_\theta)_{ij} \text{ is not frustrated}\}| \end{aligned}$$

simplifying:

$$c(\tilde{T}_\theta) = |E| + 2(|X|\cdot|Y| - |E|) + (3-1)\,|\{(i,j)\in E\colon (\tilde{\mathscr{B}}_\theta)_{ij} \text{ is frustrated}\}| + |E|$$

concluding finally the lemma:

$$c(\tilde{T}_\theta) = 2\left[\left(\frac{n-1}{3}\right)^2 + \sum_{\substack{w_{ij}=1\\ \theta_{3i,3j}\ \text{odd}}} |w_{ij}| + \sum_{\substack{w_{ij}=-1\\ \theta_{3i,3j}\ \text{even}}} |w_{ij}|\right]. \qquad \square$$

Lemma 10. *Given a semi-feasible configuration* $\tilde{T}_{\theta_1}$, *there exists a rotation* θ_2 *such that* $\tilde{T}_{\theta_2}$ *is a feasible configuration and* $c(\tilde{T}_{\theta_2}) \leqslant c(\tilde{T}_{\theta_1})$.

Proof. We proceed by transforming, for each $k\in\{1,\dots,(n-1)/3\}$, the original configuration $\tilde{T}_{\theta_1}$ into a k-feasible one without changing the interaction of the kth column of blocks and the kth row of blocks with the rest of the configuration (see Fig. 35):

$\theta = \theta_1$
For each $k\in\{1,\dots,(n-1)/3\}$

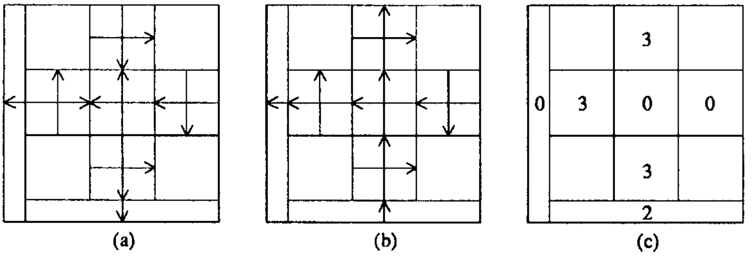


Fig. 35. (a) $\tilde{T}_\theta$ is not 2-feasible. (b) $\tilde{T}_{\theta'}$ is 2-feasible. (c) Rotations applied to the blocks.

If $\tilde{T}_\theta$ is not k-feasible
 Preserve the orientation of the diagonal block $(\tilde{B}_\theta)_{kk}$
 Rotate properly $(\tilde{\mathscr{B}}_\theta)_{ik}$ with $i \in X\backslash\{k\}$, and $(\tilde{B}_\theta)_{kj}$ with $j \in Y\backslash\{k\}$
 Rotate properly $\tilde{T}_{1,3k}$ and $\tilde{T}_{3k,1}$
$\theta_2 = \theta$.

Suppose that we obtain $\tilde{T}_{\theta'}$ applying step k to the non k-feasible configuration $\tilde{T}_\theta$. Then:

$$c(\tilde{T}_{\theta'}) - c(\tilde{T}_\theta) = c_k(\tilde{T}_{\theta'}) - c_k(\tilde{T}_\theta) + \sum_{i \in X} [c_{\text{int}}((\tilde{B}_{\theta'})_{ik}) - c_{\text{int}}((\tilde{B}_\theta)_{ik})] + \sum_{j \in Y} |c_{\text{int}}((\tilde{B}_{\theta'})_{kj}) - c_{\text{int}}((\tilde{B}_\theta)_{kj})]$$

by Lemmas 6 and 7:

$$c(\tilde{T}_{\theta'}) - c(\tilde{T}_\theta) \leqslant -20 + (3-1)\,|\{i \in X\text{: } (\tilde{B}_{\theta'})_{ik} \text{ is frustrated}\}| + (3-1)\,|\{j \in Y\text{: } (\tilde{B}_{\theta'})_{kj} \text{ is frustrated}\}|$$

it is direct that

$$c(\tilde{T}_{\theta'}) - c(\tilde{T}_\theta) \leqslant -20 + 2\,|\{i \in X\text{: } (i,k) \in E\}| + 2|\{j \in Y\text{: } (k,j) \in E\}|$$

and since each node of graph $G = (X, Y, E)$ has degree less than five, we conclude:

$$c(\tilde{T}_{\theta'}) - c(\tilde{T}_\theta) \leqslant -20 + 2(5+5) = 0.$$

On the other hand, by Lemma 8 we know that at the end of the process we obtain a feasible configuration $\tilde{T}_\theta$. $\square$

Lemma 11. *Given a feasible configuration $\tilde{T}_{\theta_1}$, there exists a rotation θ_2 such that the $\mathscr{T}_2$-configuration T_{θ_2} satisfies $c(T_{\theta_2}) = c(\tilde{T}_{\theta_1})$.*

Proof. We apply to an arbitrary feasible configuration $\tilde{T}_{\theta_1}$ in the following procedure.

- *Step* 1: Rotate in three units every s_5 tile located in a frustrated block. In other words, put the double arrows in parallel and pointing in the same direction of the double arrows of its adjacent central block tile (see Fig. 36).

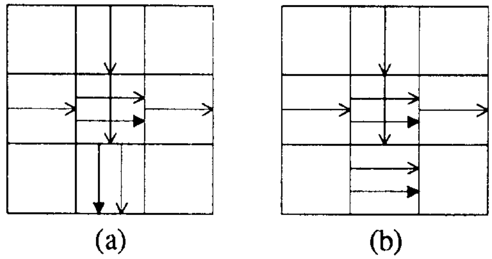


Fig. 36. (a) Frustrated block. (b) Step 1 for a frustrated block.

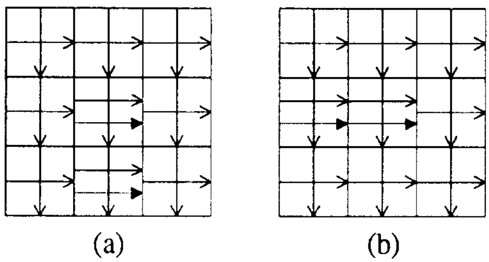


Fig. 37. (a) Step 2a for an originally frustrated block. (b) Step 2a for an originally non-frustrated block.

- *Step* 2a: For each block we add vertical and horizontal arrows (↑) to the non-central tiles in order to obtain s_1 and s_2 tiles oriented as the central one (see Fig. 37).
- *Step* 2b: Transform the s_3 and s_4 tiles of the first column and the first row into s_1 tiles oriented as their adjacent block tiles (see Fig. 38).

  With respect to previous procedure we see that:

(i) The output is a $\mathscr{T}_2$-configuration denoted by T_{θ_2}.

(ii) With first step the global cost function increases its value 20 times the number of frustrated blocks.

(iii) With second step the global cost function decreases its value 20 times the number of frustrated blocks originally frustrated. □

4.7. Equivalence between problems

Definition 11. We define the following bijection ϕ between the orientations of spin glasses (asignation of ± 1 values to the nodes of graph $G=(X,Y,E)$) and the set of feasible configurations:

$$\phi\colon ((x_i)_{i\in X},(y_i)_{i\in Y})\in\{-1,1\}^{2|X|}\mapsto \widetilde{T}_\theta, \text{ where}$$

$$\theta_{3k,3k}=\begin{cases}0 & \text{if } x_k=+1,\ y_k=+1,\\ 1 & \text{if } x_k=-1,\ y_k=+1,\\ 2 & \text{if } x_k=-1,\ y_k=-1,\\ 3 & \text{if } x_k=+1,\ y_k=-1.\end{cases}$$

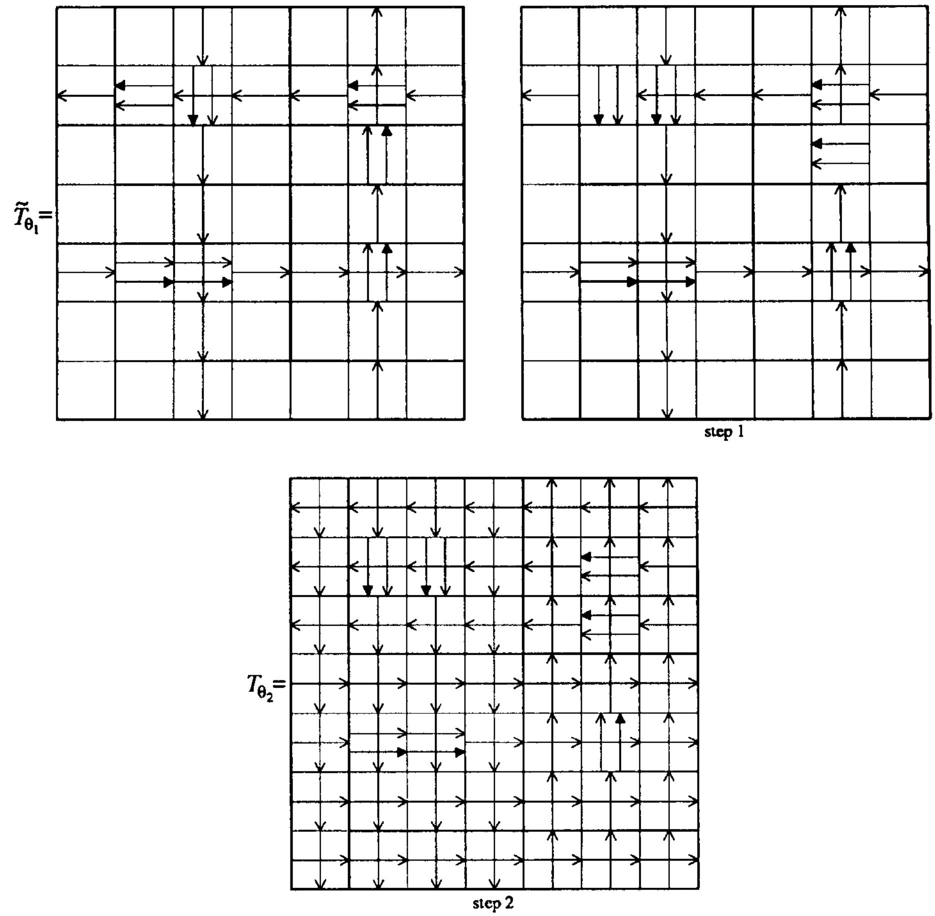


Fig. 38. Transformation of a feasible configuration into a $\mathscr{T}_2$-configuration with the same cost.

Notice that this bijection is well defined. In fact, when we fix the orientations of the central tiles of the diagonal blocks, we fix the orientation of the whole configuration (see Comments 3 and 4).

Lemma 12. *Between the quadratic product of the spin glasses and the orientations of the tiles in the feasible configurations, the following relation holds*:

$$[\exists (x,y)\in\{-1,1\}^{2|X|} \text{ such that } x_i\cdot y_j=\alpha_{ij}]$$

$$\Leftrightarrow$$

$$[\exists \theta\in\{0,1,2,3\}^{n^2} \text{ such that } \widetilde{T}_\theta \text{ is a feasible configuration and } (-1)^{\theta_{3i,3j}}=\alpha_{ij}].$$

Proof. We use bijection ϕ and we prove for the case $x_i=1\wedge y_j=-1$ (the other three are analogous).

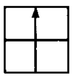
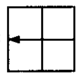
- For the case $i=j$ we have (by ϕ) $\theta_{3i,3j}=3$, and we conclude directly that $x_i \cdot y_j = (-1)^{\theta_{3i,3j}}$.
- For the case $i \neq j$:

$x_i = 1 \wedge y_j = -1$

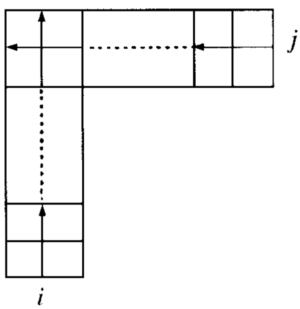
$\Leftrightarrow$

$\theta_{3i,3i} \in \{0.3\} \wedge \theta_{3j,3j} \in \{2,3\}$ (by ϕ)

$\Leftrightarrow$

$(\tilde{B}_\theta)_{ii} =$  $(\tilde{B}_\theta)_{jj} =$ 

$\Leftrightarrow$



$\Leftrightarrow$

$\theta_{3i,3j} = 3.$ □

Lemma 13. *Between the spin glasses energy and the feasible configurations cost, we have the following relation*:

$$\left[\exists (x,y) \in \{-1,1\}^{2|X|} \text{ such that } \mathscr{E}(x,y) = \sum_{(i,j)\in E} (w_{ij} x_i x_j) \geqslant k_2\right]$$

$\Leftrightarrow$

$$[\exists \theta \in \{0,1,2,3\}^{n^2} \text{ such that } \widetilde{T}_\theta \text{ is a feasible configuration and } c(\widetilde{T}_\theta) \leqslant k_1].$$

Proof. Let F be the set of feasible configurations and let $\tilde{\theta}_{ij}$ be the rotation $\theta_{3i,3j}$. By Lemma 12:

$$\mathscr{E}(x,y) \geqslant k_2 \Leftrightarrow \exists \theta \text{ such that } \widetilde{T}_\theta \in F \text{ and } \sum_{(i,j)\in E} w_{ij}(-1)^{\tilde{\theta}_{ij}} \geqslant k_2$$

$$\Leftrightarrow \exists \theta \text{ such that } \widetilde{T}_\theta \in F \text{ and}$$

$$\sum_{\substack{w_{ij}=1 \\ \tilde{\theta}_{ij} \text{ even}}} |w_{ij}| + \sum_{\substack{w_{ij}=-1 \\ \tilde{\theta}_{ij} \text{ odd}}} |w_{ij}| - \sum_{\substack{w_{ij}=1 \\ \tilde{\theta}_{ij} \text{ odd}}} |w_{ij}| - \sum_{\substack{w_{ij}=-1 \\ \tilde{\theta}_{ij} \text{ even}}} |w_{ij}| \geqslant k_2$$

$\Leftrightarrow \exists\theta$ such that $\widetilde{T}_\theta \in F$ and

$$\sum_{(i,j)\in E} |w_{ij}| - 2\left[\sum_{\substack{w_{ij}=1\\ \tilde{\theta}_{ij}\text{ odd}}} |w_{ij}| + \sum_{\substack{w_{ij}=-1\\ \tilde{\theta}_{ij}\text{ even}}} |w_{ij}|\right] \geqslant k_2$$

by definition of k_1 (see Section 4.2):

$\mathscr{E}(x,y) \geqslant k_2 \Leftrightarrow \exists\theta$ such that $\widetilde{T}_\theta \in F$ and

$$2\left[\sum_{\substack{w_{ij}=1\\ \tilde{\theta}_{ij}\text{ odd}}} |w_{ij}| + \sum_{\substack{w_{ij}=-1\\ \tilde{\theta}_{ij}\text{ even}}} |w_{ij}| + \left(\frac{n-1}{3}\right)^2\right] \leqslant k_1$$

finally, by Lemma 9:

$\mathscr{E}(x,y) \geqslant k_2 \Leftrightarrow \widetilde{T}_\theta$ is a feasible configuration and $c(\widetilde{T}_\theta) \leqslant k_1$. □

Lemma 14. *Equivalence between problems*:

$$[\exists (x,y) \in \{-1,1\}^{|2|X|} \text{ such that } \mathscr{E}(x,y) \geqslant k_2] \Leftrightarrow [\exists\theta \text{ such that } c(T_\theta) \leqslant k_1].$$

Proof. ($\Leftarrow$) Let θ belongs to $\{0,1,2,3\}^{n^2}$ such that $c(T_\theta) \leqslant k_1$.
By Lemma 3, $\exists\theta$ such that $\widetilde{T}_\theta$ is an aligned configuration and $c(\widetilde{T}_\theta) \leqslant k_1$.
By Lemma 4, $\exists\theta$ such that $\widetilde{T}_\theta$ is a semi-feasible configuration and $c(\widetilde{T}_\theta) \leqslant k_1$.
By Lemma 10, $\exists\theta$ such that $\widetilde{T}_\theta$ is a feasible configuration and $c(\widetilde{T}_\theta) \leqslant k_1$.
Finally, by Lemma 12 we conclude.

($\Rightarrow$) Let $(x,y) \in \{-1,1\}^{|2|X|}$ such that $\mathscr{E}(x,y) \geqslant k_2$.
By Lemma 12, $\exists\theta$ such that $\widetilde{T}_\theta$ is a feasible configuration and $c(\widetilde{T}_\theta) \leqslant k_1$.
By Lemma 11, $\exists\theta$ such that $c(T_\theta) \leqslant k_1$. □

By Lemma 14 (together with the facts that MTR($\mathscr{T}_2$) $\in$ NP, the NP-completeness of SG, and the polynomiality of the reduction) we conclude that MTR($\mathscr{T}_2$) is NP-complete. □

4.8. Polynomial subproblems

In the following three theorems we introduce polynomial sub-problems by restricting two MTR-parameters: the number of admissible tiles and the region where tile assignments are defined on.

Theorem 4. *MTR($\mathscr{T}$) is a polynomial problem for any singleton set of admissible tiles $\mathscr{T}$.*

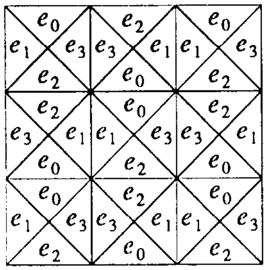


Fig. 39. Perfect tiling with one tile.

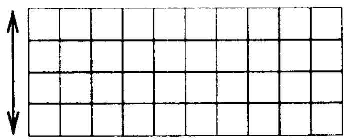


Fig. 40. Fixed width rectangle.

Proof. When $|\mathscr{T}| = 1$ it is trivial to obtain a perfect tiling (see Fig. 39). □

Theorem 5. MTR($\mathscr{T}$) *is a polynomial problem when tile assignments are defined on fixed width rectangles (for any set of admissible tiles* $\mathscr{T}$*).*

The proof appears in [10] and, as in the one-dimensional case (see [9]), it simply consists to reduce MTR into the shortest path problem (Fig. 40).

Considering a $\mathbb{Z}^2$-region as a graph in which its adjacent cells (or nodes) are connected by arcs, it follows:

Theorem 6. MTR($\mathscr{T}$) *is a polynomial problem when tile assignments are defined on acyclic regions (for any set of admissible tiles* $\mathscr{T}$*).*

Proof. The following algorithm solves the problem:

- *Initialization*: Consider an arbitrary node (of the region where the tile assignment is defined on) as a root and visualize the graph as a tree (see Fig. 41).
  Associate to each node v the *orientation-cost* vector $(v_0, v_1, v_2, v_3) \in \mathbb{N}^4$ and initialize it with $(0,0,0,0)$. That means, if we denote v_i at instant t as $v_i(t)$, we have for all $v \in V : v_0(0) = v_1(0) = v_2(0) = v_3(0) = 0$.
- *Iteration*: The idea of the algorithm is to obtain a single node tree by eliminating systematically the leaves.
  Let $l \in V$ be a leaf and $f \in V$ its father at instant t. Before eliminating l we must transfer the information it contains as follows:

$$f_i(t+1) = f_i(t) + \min_{j \in \{0,1,2,3\}} \{l_j(t) + C_{ij}(f,l)\}, \quad i = 0,1,2,3,$$

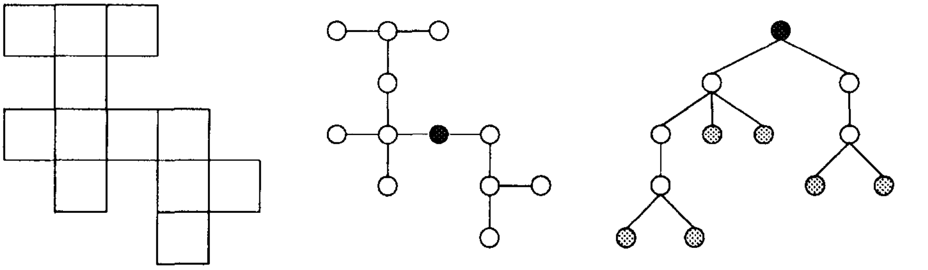


Fig. 41. (a) Region where the tile assignment is defined on. (b) Acyclic graph. (c) Associated tree.

where $C_{ij}(f,l)$ is the interaction cost between the tiles located on nodes f and l when their orientations are i and j, respectively.

Notice that the relevant information is always stored in the leaves. In fact, for a leaf l, the value l_i corresponds to the following: if the orientation of the tile located on l is i then the minimal cost of the configuration that only includes its descendants (already eliminated) is l_i.

Previous algorithm stops when it reaches a tree with a single-node r. Therefore, there exists a configuration whose cost is less than k if and only if $\min\{r_0,r_1,r_2,r_3\}\leqslant k$.

Finally, due to the fact that the number of steps required by the algorithm is proportional to the number of nodes, we conclude that it solves the problem in polynomial time. □

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