Small Alliances in Graphs ⋆

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Abstract. Let $G = (V, E)$ be a graph. A nonempty subset $S \subseteq V$ is a (strong defensive) *alliance* of G if every node in S has at least as many neighbors in S than in $V \setminus S$. This work is motivated by the following observation: when G is a locally structured graph its nodes typically belong to small alliances. Despite the fact that finding the smallest alliance in a graph is NP-hard, we can at least compute in polynomial time $depth_G(v)$, *the minimum distance one has to move away from an arbitrary node v in order to find an alliance containing v.*

We define $depth(G)$ as the sum of $depth_G(v)$ taken over $v \in V$. We prove that $depth(G)$ can be at most $\frac{1}{4}(3n^2 - 2n + 3)$ and it can be computed in time $O(n^3)$. Intuitively, the value $depth(G)$ should be small for clustered graphs. This is the case for the plane grid, which has a depth of $2n$. We generalize the previous for bridgeless planar regular graphs of degree 3 and 4.

The idea that clustered graphs are those having a lot of small alliances leads us to analyze the value of $r_p(G) = \mathbb{P}\{S \text{ contains an alliance}\}$, with $S \subseteq V$ randomly chosen. This probability goes to 1 for planar regular graphs of degree 3 and 4. Finally, we generalize an already known result by proving that if the minimum degree of the graph is logarithmically lower bounded and if S is a large random set (roughly $|S| > \frac{n}{2}$), then also $r_p(G) \to 1$ as $n \to \infty$.

1 Introduction

The clustering coefficient of a vertex v, denoted by $c(v)$, indicates the extent to which neighbors of v are neighbors themselves [1]. More precisely, if the number of edges within the neighborhood of v is Γ and the degree of v is d, then $c(v) = \frac{2\Gamma}{d(d-1)}$. The average of $c(v)$ taken over all the nodes of a graph G gives the clustering coefficient of G. With this coefficient, Watts and Strogatz [1] were able to justify empirically the idea that small-world networks are locally connected while classical random graphs are not (with both families having a small diameter).

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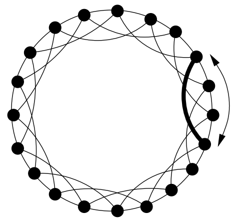

Fig. 1. A locally structured graph.

A fundamental limitation of the clustering coefficient is shown in the graph of Fig. 1, consisting of a ring where every vertex has two additional neighbors at ring-distance k. When $k = 2$, the clustering coefficient is $\frac{1}{2} > 0$. However, when $k \geq 3$ it becomes 0, despite the fact that the structure of the graph remains the same.

This and other limitations have lead researchers to propose all kinds of generalizations. In [2], for instance, authors introduced a new definition intended to filter out the effect of degree correlations. In [3], instead of asking "how many of my neighbors are connected?", researchers started to ask "how closely related are my neighbors?". Roughly, this is the approach behind most of the new notions such as grid coefficient [4], meshedness coefficient [5], weighted clustering coefficient [6], high order clustering coefficient [7] and efficiency [8].

In the present work we take another approach, which is motivated by the following observation: the nodes of locally structured graphs typically belong to small *alliances*. In Fig. 1, these alliances are cycles of length k. More precisely, a subset of nodes $S \subseteq V$ is an alliance if each of its nodes has at least as many neighbors inside S than outside S. The formal definition was given in [9], where they used the term *strong defensive alliance* (which for simplicity we call alliance).

Our results. Let S_v be, among all the alliances containing the node v, a minimum one (with respect to the cardinality). Since finding $|S_v|$ is NP-hard, we exhibit in Sect. 2 a polynomial time algorithm that computes $depth_G(v)$, the minimum distance one has to move away from v in order to find an alliance containing v. We define $depth(G)$ as the sum of $depth_G(v)$ taken over all the nodes of G. We prove that the depth of G can be at most $\frac{1}{4}(3n^2 - 2n + 3)$ and it can be computed in time $O(n^3)$. For *bridgeless planar* cubic graph we obtain a better upper bound, namely $\frac{15}{2}n$. We consider this section the starting point of our research efforts by which we expect to find bounds for the depth of different graph classes.

We also take a probabilistic approach introducing another coefficient. We consider the probability of finding an alliance in a randomly chosen subset of nodes S (each one independently and with probability p). We prove that, as expected and in accordance with the previous result, this probability goes to 1 for planar regular graphs of degree 3 and 4.

It is known that in every graph $G = (V, E)$ of n nodes there exists an alliance of size at most $\lfloor \frac{n}{2} \rfloor + 1$ [9, 10]. We prove a stronger result which says that, for graphs where the degree of every node is $\omega(\log(n))$, if the chosen set S is large enough (i.e, with $p > \frac{1}{2}$), then S will be an alliance with high probability.

Related work. The notion of alliance was first studied in [9], where the authors introduced various types of alliances which have been studied later, calculating and bounding their size on certain classes of graphs. Namely, these types of alliances are: defensive alliances [9, 11], offensive alliances [12], global defensive/offensive alliances [13, 14], dual or powerful alliances [15] and k-alliances [16–19].

The notion of alliance is very natural and, for that reason, it has appeared in other works in different contexts. In [20] the notion of *web community* was introduced: "a community is a set of sites that have more links to members of the community than to non-members". In [21] the authors refer to a "white block" as a subset W of an $(m \times n)$-torus composed of vertices "each of which has at least two neighbors in W". This set W is, of course, an alliance. It appeared when researchers were trying to bound the size of monopolies and coalitions in graphs [22, 23]. A closely related line of research consists in trying to partition the graph into communities (alliances in this work). Here the key object is the partition itself and the measure of its quality. Newman in [24], together with a state-of-the-art survey and a complete list of references, provides an algorithm for partitioning based on the eigenspectrum of a matrix he calls modularity matrix.

Some terminology. Let $G = (V, E)$ be a (simple) undirected graph. We will usually assume $|V| = n$. Let $X \subseteq V$ and $v \in V$. Let $d_X(v)$ be the number of neighbors the node v has in X. In other words, $d_X(v) = |N_G(v) \cap X|$, where $N_G(v)$ is the (open) neighborhood of v. A nonempty subset $S \subseteq V$ is a strong defensive alliance [9] if for every vertex $v \in S$ it holds that $|N_G(v) \cap S| \geq |N_G(v) \cap \overline{S}|$. Note that this is equivalent to $d_S(v) \geq d_{\overline{S}}(v)$. In this work such a set S will simply be called an alliance. The eccentricity of a node v, denoted by $ecc_G(v)$, is the greatest distance between v and any other node in G.

2 The Depth of a Graph

Let v be a node of a graph $G = (V, E)$. Let $S_v \subseteq V$ denote a minimum size alliance containing v. Our work is motivated by the following observation: in locally structured graphs the value $|S_v|$ is typically small. Therefore, if we want to measure how locally structured a graph is, we should compute the average of $|S_v|$ taken over all the nodes. Unfortunately, calculating the size of each S_v turns out to be NP-hard. In fact, let us define the problem ALLIANCE as follows:

ALLIANCE
Instance: Graph G and $k \in \mathbb{N}$.
Question: Is there any alliance S in G such that $|S| \leq k$?

This problem is NP-complete [25]. For sake of completeness we present our own reduction in the Appendix. Despite the fact that the previous result implies that in practice *there is no efficient way to find* $|S_v|$, we can still do something. In fact, since we are looking for a measure of "clustering", it would be enough to compute the *depth* of v, the minimum distance one has to move away from v in order to find an alliance containing v:

$$depth_G(v) = \min\{ecc_S(v) : S \text{ alliance with } v \in S\},$$

where $ecc_S(v)$ is the S-eccentricity of v, the distance from v to the farthest node in S. We are going to present first an algorithm that, given $A \subseteq V$, outputs $m(A)$, the largest alliance contained in A.

ALLIANCE Input: $G = (V, E), A \subseteq V$. Output: $m(A)$.
$S \leftarrow A$
$S' \leftarrow \{v \in S : 2d_S(v) \geq d_G(v)\}$
while $S' \neq S$ **do**
  $S \leftarrow S'$
  $S' \leftarrow \{v \in S : 2d_S(v) \geq d_G(v)\}$
end while
return S

Proposition 1. *If the set $m(A)$ computed by algorithm **ALLIANCE** is not empty, then it is the largest alliance contained in A. The time complexity of **ALLIANCE** is $O(n^2)$.*

Proof. Let S be any set of vertices and let

$$S' = \{v \in S : 2d_S(v) \geq d_G(v)\}$$

Clearly, $S' = S$ if and only if S is an alliance. Moreover, an alliance is contained in S if and only it is contained in S'. Hence, the largest alliance contained in S (if any) is also contained in S'. Therefore, if **ALLIANCE** sets S' to $\emptyset$ during some iteration, then it will finish with $m(A) = \emptyset$. Otherwise, it stops with $m(A) = S' = S \neq \emptyset$, for some set S. For the time complexity notice that the construction of S' is $O(n)$ and there are at most n iterations. □

By using **ALLIANCE** we propose the following algorithm to compute the depth of a vertex.

DEPTH Input: $G = (V, E), v \in V$. Output: $depth_G(v)$.
$A \leftarrow N_G(v) \cup \{v\},\ r \leftarrow 1$
while $r \leq n$ **do**
  if $v \in$ **ALLIANCE**(A) **then**
    return r
  end if
  $A \leftarrow A \cup N_G(A)$
  $r \leftarrow r + 1$
end while

Proposition 2. ***DEPTH*** *returns $depth_G(v)$ and its time complexity is $O(n^3)$.*

Proof. In order to prove the statement we prove that the depth of a vertex v corresponds to the smallest radius $r > 0$ such that the ball of radius r centered in v contains an alliance containing v, which is exactly the quantity returned by **DEPTH**.

Clearly, if S is an alliance contained in a ball of radius r centered in v, then the distance between v and any vertex in S is at most r. Hence the eccentricity of v in S is at most r. Therefore, the depth of v is at most r. Conversely, for sake of contradiction, let us assume that there is an alliance S containing v such that the eccentricity of v in S is less than r. Then the distance from v to any vertex in S is less than r. Hence, S is an alliance contained in a ball of radius smaller than r.

Since running **DEPTH** involves running **ALLIANCE** at most n times, the time complexity follows. □

The depth of a graph G is the sum of the depth of its vertices. It is denoted by $depth(G)$. From Proposition 2, $depth(G)$ can be computed in polynomial time. As we have already mentioned, it is known that every graph G with n vertices has an alliance of size at most $\left\lfloor \frac{n}{2} \right\rfloor + 1$ [9, 10]. In order to find an upper bound for the depth of G we prove now a slightly different result.

Proposition 3. *Every graph $G = (V, E)$ has an alliance $S \subseteq V$ such that $|S| \in \{\lceil \frac{n}{2} \rceil,\ \lceil \frac{n}{2} \rceil + 1\}$.*

Proof. Let us consider the set of all "almost balanced" cuts of $G = (V, E)$. More precisely, $\mathcal{C} = \left\{E(U, \overline{U}) \subseteq E\ :\ U \subseteq V,\ |U| \in \{\lceil n/2 \rceil,\ \lceil n/2 \rceil + 1\}\right\}$. Let $U_0 \subseteq V$ be such that $E(U_0, \overline{U_0})$ is a min-cut of $\mathcal{C}$. i.e, $|E(U_0, \overline{U_0})| = \min_{\tilde{E} \in \mathcal{C}} |\tilde{E}|$.

All we need to prove now is that U_0 is an alliance of G. Suppose that there is a node $u \in U_0$ such that $d_{U_0}(u) < d_{\overline{U_0}}(u)$. In that case, if we define $U_1 = U_0 \backslash \{u\}$, we would have

$$|E(U_1, \overline{U_1})| = |E(U_0, \overline{U_0})| - (d_{\overline{U_0}}(u) - d_{U_0}(u)) < |E(U_0, \overline{U_0})|.$$

In order to conclude we need to show that $E(U_1, \overline{U_1}) \in \mathcal{C}$. In fact, if $|U_0| = \lceil n/2 \rceil + 1$ then $|U_1| = \lceil n/2 \rceil$. On the other hand, if $|U_0| = \lceil n/2 \rceil$ then $|\overline{U_1}| \in \{\lceil n/2 \rceil,\ \lceil n/2 \rceil + 1\}$. □

Corollary 1. *The depth of any graph G is at most $\frac{1}{4}(3n^2 - 2n + 3)$.*

Proof. The depth of $\left\lceil \frac{n}{2} \right\rceil$ vertices is at most $\left\lceil \frac{n}{2} \right\rceil$. □

We do not know whether this upper bound is tight. Nevertheless, by forcing the graph to be bridgeless planar of degree at most 4, the upper bound decreases drastically.

First notice the following: the depth of every vertex in the $(m \times n)$-grid is 2, since every vertex belongs to a small alliance (a cycle of length 4). We are now going to generalize this and prove that the depth of any planar bridgeless graph of degree at most 4 is linear in n. This result goes in the right direction: the depth of a graph seems to be a good generalization of its clustering coefficient.

Let G be a bridgeless plane cubic graph. Then, each vertex v belongs to a cycle C which is the boundary of a face. We call these faces *facial cycles* in the sequel. Clearly, C is an alliance containing v, and therefore $depth_G(v) \leq |V(C)|/2$.

Proposition 4. *The depth a bridgeless planar cubic graph is at most* $\frac{15}{2}n$.

Proposition 4 is a consequence of the following lemma.

Lemma 1. *Let* $G = (V, E)$ *be a bridgeless plane cubic graph and let* $F(G)$ *denote the set of its faces. Then there exists a function* f *associating to each vertex a facial cycle containing it and such that no face is associated with more than five vertices.*

Proof. Consider the dual graph $G^* = (F(G), E^*)$. Since G is plane, cubic and bridgeless, then G^* is a plane graph with no loops and with no multiple edges. We are looking for a function f that assigns to each vertex of G a particular alliance to which it belongs. By duality, this is equivalent to look for a function f^* that assigns to each face h^* of G^* a particular vertex v^* of G^* with v^* lying in the boundary of h^*. Hence, in order to prove the lemma we have to make sure that in our construction (of f^*) at most 5 faces of G^* are labeled with the same vertex.

We proceed by induction on the number of vertices of G^*. If G^* has just one vertex then we label the unique face of the graph with this vertex. Suppose now that G^* has $n+1$ vertices. Since G^* is plane (with no loops and with no multiple edges) there must be a vertex v^* of degree at most five. Consider the graph $G' = G^* \setminus v^*$ (i.e, we delete the vertex and the incident edges). By the induction hypothesis one can solve the problem in G' without using v^*. The point occupied by v^* belongs to one face of G'. That face contains at most 5 faces of G^*. We label all of them with v^* and we get the result. $\square$

Proof. (of Proposition 4). Let us consider the function f of the previous lemma. It follows:

$$\sum_{v \in V} |f(v)| = \sum_{h \in F(G)} |h||f^{-1}(h)| \leq 5 \sum_{h \in F(G)} |h| = 5 \times 2|E| = 5 \times 3|V|.$$

Therefore, $depth(G) = \sum_{v \in V} depth_G(v) \leq \frac{1}{2} \sum_{v \in V} |f(v)| \leq \frac{15}{2}|V|$. $\square$

3 A Probabilistic Approach

Clustered graphs are those having a lot of small alliances. So a natural way of testing this is to compute the probability of finding an alliance in a small fraction of nodes (chosen randomly).

We can formalize this question. Let $p \in [0, 1]$. Let us denote $V_p(G)$ the outcome of selecting each node of V with probability p. Let us denote $r_p(G) = \mathbb{P}\{V_p(G) \text{ contains an alliance}\}$. The problem of computing $r_p(G)$ seems to be very difficult in general. But it can be tackled in some cases.

Proposition 5. *Let $G = (V, E)$ be a cubic planar graph. Let $0 < p < 1$. Then $r_p(G) \geq 1 - (1 - p^6)^{\frac{n+4}{56}}$.*

Proof. Let us assume that G is already embedded in the plane and let F be the set of faces of G. As any face is an alliance of G, we have that

$$r_p(G) \geq \mathbb{P}\{S : \exists f \in F, V(f) \subseteq S\}.$$

Let F' be any maximal set of vertex pairwise disjoint faces of size at most 6. The probability that a random set S does not contain a given face f in F' is $1 - p^{|V(f)|} \leq 1 - p^6$. Since the faces in F' are vertex disjoint the probability that S does not contain any face in F' is at most $(1 - p^6)^{|F'|}$. In order to conclude we will prove that $56|F'| \geq |V| + 4$.

Let a_i be the number of faces of size i and let b_i be the number of faces of size at most i. By maximality, any face f with size at most 6 intersects at least one element of F' and a face $f \in F'$ intersect at most 6 faces of size at most 6 not in F'. Therefore, $6|F'| \geq b_6 - |F'|$ and then $7|F'| \geq b_6$. From the definition of a_i, we get that $|F| = \sum_{i\geq 3} a_i$ and $2|E| = \sum_{i\geq 3} ia_i$. From Euler's Formula for cubic graphs, $2|E| = 6|F| - 12$, we get the following.

$$\sum_{i\geq 7}(i - 6)a_i + 12 = a_5 + 2a_4 + 3a_3 \tag{1}$$

Let c be a positive number. From equation 1 we deduce that if $a_5 + 2a_4 + 3a_3 < c|F|$, then $|F| - b_6 < c|F|$ and hence $b_6 > (1 - c)|F|$. Otherwise, if $a_5 + 2a_4 + 3a_3 \geq c|F|$ then $b_6 \geq \frac{c}{3}|F|$. By choosing $c = \frac{3}{4}$ we conclude that $b_6 \geq \frac{1}{4}|F|$. By using again Euler's formula and the upper bound $|F'| \geq \frac{b_6}{7}$ we conclude that

$$|F'| \geq \frac{b_6}{7} \geq \frac{1}{28}|F| = \frac{1}{28}(2 + |V|/2)$$

Therefore,

$$r_p(G) \geq 1 - (1 - p^6)^{|F'|} \geq 1 - (1 - p^6)^{\frac{|V|+4}{56}}.$$

□

We say that a sequence of graphs $(G_k)_{k\in\mathbb{N}}$ is an increasing sequence if the order (number of nodes) of the graphs grows with k.

Corollary 2. *Let $0 < p < 1$. Every increasing sequence $(G_k)_{k\in\mathbb{N}}$ of cubic planar graphs satisfies $\lim_{k\to\infty} r_p(G_k) = 1$.*

Remark 1. The previous result also holds for planar regular graphs of degree 4.

As we have already seen, every graph G with n nodes has an alliance of size at most $\left\lfloor \frac{n}{2} \right\rfloor + 1$. This alliance comes from a *very particular construction*, dealing with an "almost balanced" minimum cut of G. What if we choose randomly a large set of nodes? Is it going to contain an alliance with high probability?

Proposition 6. *Let $G = (V, E)$ be a graph with minimum degree d. Let $\frac{1}{2} < p < 1$. Then $r_p(G) \geq 1 - ne^{-\frac{p\delta^2}{2}d}$, where $\frac{1}{2} = p(1-\delta)$.*

Proof. We apply the Chernoff bound in a standard way as explained in [26]. Let $X_v = 1$ if $v \in V_p(G)$ and $X_v = 0$ otherwise. Let $X(v) = \sum_{u \in N(v)} X_u$. It follows:

$$\begin{aligned} r_p(G) &\geq \mathbb{P}\{\forall v \in V_p(G) : d_{V_p(G)}(v) \geq d_{\overline{V_p(G)}}(v)\} \\ &\geq 1 - \sum_{v \in V_p(G)} \mathbb{P}\{X(v) < \frac{d(v)}{2}\} = 1 - \sum_{v \in V_p(G)} \mathbb{P}\{X(v) < (1-\delta)\mathbb{E}(X(v))\} \\ &\geq 1 - \sum_{v \in V_p(G)} e^{-\frac{p\delta^2}{2}d(v)} \geq 1 - ne^{-\frac{p\delta^2}{2}d}. \end{aligned}$$

□

We can apply the previous lemma to graphs for which the degree of every node is high enough. A class of graphs is said to have minimum degree $d(n)$ if the minimum degree of any graph having more than n nodes is at least $d(n)$.

Corollary 3. *Let $\frac{1}{2} < p < 1$ and let $d(n) = \omega(\log(n))$. Then, for every increasing sequence $(G_k)_{k \in \mathbb{N}}$ of graphs with minimum degree $d(n)$, we have $\lim_{k \to \infty} r_p(G_k) = 1$.*

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Appendix: Alliance is NP-complete

It is known that the following problem is NP-complete [27].

> HALF-CLIQUE
> **Instance:** Graph G (with n nodes, n even).
> **Question:** Is there any clique in G of size (at least) $\frac{n}{2}$?

Proposition 7. Alliance *is NP-complete.*

Proof. Let $G = (V, E)$ be an instance of Half-Clique (i.e, n is even). In the reduction to Alliance we construct a graph G' of size $4n$ as follows. We first generate graphs $G_1 = (V, E)$, $G_2 = (V, \phi)$, $G_3 = (V, \phi)$ and $G_4 = (V, E)$. The connection between these graphs is made according to the original graph $G = (V, E)$. Here we will abuse the notation, making no distinction between the copies in the four graphs, of a node $v \in V$.
Nodes $t \in G_1$ are connected to those nodes $u \in G_2$ such that $t \in V \setminus (\{t\} \cup N(u))$. Nodes $u \in G_2$ are connected to those nodes $v \in G_3$ such that $u \in N(v)$. Nodes $v \in G_3$ are connected to those nodes $w \in G_4$ such that $v \in (V \setminus (\{v\} \cup N(w))$. For each $i \in \{1, 2, 3, 4\}$, v_i is connected to every vertex of G_i. Notice that G' is n-regular and therefore the smallest alliances are of size $\frac{n}{2}$. The reader should verify that there exists an alliance in G' of size $\frac{n}{2} + 1$ if and only if there exists a clique in G of size $\frac{n}{2}$. $\square$