On dissemination thresholds in regular and irregular graph classes ⋆

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Abstract We investigate the natural situation of the dissemination of information on various graph classes starting with a random set of informed vertices called active. Initially active vertices are chosen independently with probability p, and at any stage in the process, a vertex becomes active if the majority of its neighbours are active, and thereafter never changes its state. We show that in any cubic graph, with high probability, the information will not spread to all vertices in the graph if $p < \frac{1}{2}$. We give families of graphs in which information spreads to all vertices with high probability for relatively small values of p.

1 Introduction

Let $G = (V, E)$ be a simple undirected graph. A *configuration* C of G is a function that assigns to every vertex in V a value in $\{0, 1\}$. The value 1 means that the corresponding vertex is *active* while the value 0 represents *passive* vertices.

We investigate the natural situation in which a vertex v needs a *strong majority* of its neighbours, namely strictly more than $\frac{1}{2}d(v)$ neighbours, to be active in order to become an active vertex. Therefore, consider the following rule of *dissemination* that acts on configurations: a passive vertex v whose strict majority of neighbours are active becomes active; once active, a vertex never changes its state. The initial configuration of a dissemination process is called an *insemination.* Since the set of active vertices grows monotonically in a finite set V, a fixed point has to be reached after a finite number of steps. If the fixed point is such that all vertices have become active, then we say that the initial configuration *overruns* the graph G. A *community* [10] (also called an *alliance*) in G is a subset of nodes $X \subseteq V$ each of which has at least as many neighbours in X as in $V \setminus X$,

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i.e. for every $v \in X$, $|N(v) \cap X| \geq |N(v) \cap (V \setminus X)|$. Notice that a configuration overruns G if and only if it contains no community of passive vertices.

Dissemination has been intensively studied in the literature, using various dissemination rules (see e.g. [17] for a survey). Among other types of rules we can cite models in which a vertex becomes active if the total weight of its active neighbours exceeds a fixed value [12], or symmetric majority voting rules, for which an active vertex may also become passive if the number of passive neighbours outweights the number of active neighbours [17]. One of the main questions for each of these models is to find small sets of active vertices which overrun the network. Several authors considered the problem of finding small communities in arbitrary graphs or special graph classes [8,10].

In this work we consider a probabilistic framework. A random configuration in which each vertex is active with probability p and passive with probability $1-p$ is called a *p-insemination*. We are interested in the probability $\theta_p(G)$ that a p-insemination overruns G. It is clear that $\theta_p(G)$ is a monotonic increasing function of p. We investigate the majority dissemination process starting with a p-insemination for various graph classes. Such random dissemination processes, with different types of dissemination rules, have been studied in the literature in the context of cellular automata or in bootstrap percolation [11].

One of the basic questions is to determine the ratio of active vertices (in other words, the critical value of p) one needs in order to overrun the whole graph with high probability. Without any restriction on the structure of the underlying graph, it appears to be difficult to determine this ratio. It is therefore more instructive to consider whole classes of graphs. If $\mathcal{G}$ is a class of graphs, let $\boldsymbol{G} = (G_n)_{n\in\mathbb{N}}$ denote a generic sequence of graphs $G_n \in \mathcal{G}$ such that $|V(G_n)| < |V(G_{n+1})|$ for all $n \in \mathbb{N}$. We define *dissemination half-thresholds* p_c^+ and p_c^- of class $\mathcal{G}$ by

$$p_c^+(\mathcal{G}) = \inf\{p \mid \exists \boldsymbol{G} : \lim \theta_p(G_n) = 1\}$$
$$p_c^-(\mathcal{G}) = \sup\{p \mid \forall \boldsymbol{G} : \lim \theta_p(G_n) = 0\}$$

In words, for $p < p_c^-$ and any increasing sequence $\boldsymbol{G}$ in $\mathcal{G}$, the probability that a random p-insemination overruns the graph tends to zero.

For example, for the class $\mathcal{K}$ of all complete graphs, it is straightforward to see that $p_c^+(\mathcal{K}) = p_c^-(\mathcal{K}) = \frac{1}{2}$. If for a class $\mathcal{G}$ the two half-thresholds are equal, we say that $p_c(\mathcal{G}) = p_c^+(\mathcal{G}) = p_c^-(\mathcal{G})$ is the *dissemination threshold* of class $\mathcal{G}$. It is convenient to introduce the following terminology: throughout the paper, if $(A_n)_{n\in\mathbb{N}}$ is a sequence of events in a probability space such that $\lim_{n\to\infty} \mathbb{P}[A_n] = 1$, we write A_n a.a.s (asymptotically almost surely). For example, if $p < p_c^-(\mathcal{G})$ then a.a.s $G_n \in \mathcal{G}$ is not overrun by a p-insemination.

In this paper, we consider dissemination on regular graphs and particular classes of irregular graphs. First we consider regular graphs, for which we give simple lower bounds for the dissemination half-threshold p_c^-, and we prove that the threshold p_c is exactly $\frac{1}{2}$ for cubic graphs. In the second part, we give simple explicit constructions of graph classes with relatively small dissemination half-threshold $p_c^+(\mathcal{G})$. This counters the naive intuition that one should need about half of the vertices to overrun the whole graph.

Regular graphs. The dissemination process, as we have mentioned, has been studied for specific families of graphs, such as integer lattices, hypercubes, and so on, all of which are regular graphs. More generally, let $\mathcal{G}_r$ be the family of r-regular graphs. We observe that $p_c(\mathcal{G}_2) = 1$, since a p-insemination overruns a cycle if and only if there are no two consecutive passive vertices. A more interesting case is the class $\mathcal{Q}$ of hypercube graphs: these are regular graphs but with growing degrees. Following from more general results on families of regular graphs with growing degrees, Balogh, Bollobás and Morris [5] showed $p_c(\mathcal{Q}) = \frac{1}{2}$. Balogh and Pittel [6] considered dissemination on random r-regular graphs. Consider $G_{n,r}$, a graph chosen uniformly at random from the family of all r-regular graphs on n vertices, so $\mathcal{G}(r) = \{G_{n,r} : n \in \mathbb{N}\}$. It turns out that $p_c(\mathcal{G}(r))$ a.a.s. exists and equals

$$p_r := 1 - \inf_{y \in (0,1)} \frac{y}{F(r-1, 1-y)}$$

where $F(r, y)$ is the probability of obtaining at most $r/2$ successes in r independent trials with the success probability equal y. This leaves the determination of the dissemination threshold for $\mathcal{G}_r$ for fixed $r > 3$ as an open question. Let us have a look at the values of p_r for small r, though:

r	3	4	5	6	7
p_r	0.5	0.667	0.275	0.397	0.269

We conjecture that the dissemination thresholds $p_c(\mathcal{G}_r)$ exist and equal p_r. Towards this conjecture, we show the following modest result:

Theorem 1. *For all positive integers r, $p_c^-(\mathcal{G}_r) \leq p_r$ and*

$$p_c^-(\mathcal{G}_r) \geq \begin{cases} \frac{1}{r} & \text{if } r \text{ is odd} \\ \frac{2}{r} & \text{if } r \text{ is even} \end{cases}$$

We will prove the conjecture in the case $r = 3$:

Theorem 2. $p_c(\mathcal{G}_3) = \frac{1}{2}$

Irregular graphs. It is natural to search for graph classes $\mathcal{G}$ for which $p_c^+(\mathcal{G})$ is small. If, as we conjecture, regular graphs behave like random regular graphs, then regular graphs cannot have very low thresholds. One should consider graphs whose vertices have varying degrees – we refer to these loosely as irregular graphs. To this end, we consider the class of wheels and toroidal graphs. Let C_n denote the cycle on n vertices and C_n^2 denote the the toroidal grid on n^2 vertices. Notice that C_n^2 is, indeed, the cartesian square of C_n. In general, let C_n^k denote the k-dimensional torus. Let $u * C_n^k$ denote the k-dimensional torus augmented with a single universal vertex u. We will consider the class of wheels – i.e. the family $\mathcal{W} = \{u{*}C_n \mid n \in \mathbb{N}\}$ – and the class of toroidal grids plus a universal vertex – i.e. $\mathcal{T} = \{u * C_n^2 \mid n \in \mathbb{N}\}$. Our main result is that for both classes the dissemination threshold is small:

Theorem 3. *For the class $\mathcal{W}$, we have $p_c^+(\mathcal{W}) = 0.4030...$, where $0.4030...$ is the unique root in the interval $[0, 1]$ of the equation $p + p^2 - p^3 = \frac{1}{2}$. For the class $\mathcal{T}$ of toroidal grids plus a universal vertex, we have $0.35 \leq p_c^+(\mathcal{T}) \leq 0.372$.*

Since our goal is to find graph classes with small dissemination thresholds, clearly the second result is stronger than the first. Nevertheless, we shall present their proofs in parallel. For establishing the bounds on toroidal grids plus a universal vertex we need (a small amount of) computer-aided computations, while on wheels all computations are easy to check by hand.

The results of Balogh and Pittel on 7-regular graphs imply the existence of graph classes with half-threshold $p_c^+ < 0.27$. Although this bound is smaller than in our case, our result has the advantage of giving explicit constructions of graph classes with small half-threshold p_c^+. We also believe that our proof techniques might give new tools for constructing classes with even smaller values of p_c^+. Let us remark that computer simulations for higher dimension tori with a universal vertex $u * C_n^k$ indicate even lower thresholds. In simulations, a random p-insemination overruns $u * C_n^2$ the graph a.a.s. already with $p = 0.37$, which fits within the bounds shown in this paper. For k equal 3, 4 and 5 the graph $u * C_n^k$ is a.a.s. overrun by a random p-insemination already with p equal 0.35, 0.32 and 0.3, respectively. We leave the following as an open problem: Is there a family of graphs on which any p-insemination overruns the graph a.a.s for any $p > 0$?

2 Regular graphs

In this section we outline the proof of Theorem 1. Balogh and Pittel [6] showed that for the class of random r-regular graphs, the dissemination threshold is a constant p_r a.a.s. where $p_3 = \frac{1}{2}$, $p_4 = \frac{2}{3}$ and so on. This establishes the upper bound in Theorem 1. For the lower bound, we use the following easy observation. The average degree of a graph G is $2e(G)/|V(G)|$.

Lemma 1. *Let G be a graph of average degree more than $2k - 2$, where $k \in \mathbb{N}$. Then G has a subgraph of minimum degree at least k.*

Proof. Let G be such a graph. We recursively remove vertices of degree at most $k - 1$. Each step this removes at most $k - 1$ edges, thus at the end of this process we must obtain a non-empty subgraph of G. This subgraph has the required property. □

Let I be the set of active vertices of a p-insemination of $G \in \mathcal{G}_r$, and $I^c = V(G) \setminus I$. Then

$$\mathbb{E}[|I^c|] = (1-p)n \quad \text{and} \quad \mathbb{E}[e(I^c)] = \frac{r}{2}(1-p)^2 n$$

where $e(I^c)$ is the number of edges of G with both ends in I^c. Note that $|I^c|$ is a binomial random variable, in particular the the Chernoff Bound [2] implies:

$$|I^c| \sim (1-p)n \quad \text{a.a.s.} \tag{2.1}$$

We also need to prove that

$$e(I^c) \sim \frac{r}{2}(1-p)^2 n \quad \text{a.a.s.} \tag{2.2}$$

This is proved using the Independent Bounded Differences (IBD) inequality (see [14]).

Theorem 4 ([14]). *Let $X = (X_1, X_2, \ldots, X_q)$ be a family of independent random variables with X_i taking values in a set A_i for each i. Suppose that the real-valued function f defined on ΠA_i satisfies*

$$|f(x) - f(x')| \leq c_i$$

whenever vectors x and x' only differ on the ith coordinate. Let μ be the expected value of $f(X)$. Then for any $t \geq 0$,

$$\mathbb{P}(|f(X) - \mu| \geq t) \leq 2e^{-2t^2/\sum c_i^2}.$$

Note that $e(I^c)$ can be considered as a function of the independent variables X_v, for all vertices v of the graph, where $X_v = 1$ if v is active in the initial configuration, and $X_v = 0$ if v is initially passive. By changing the value of only one variable X_v, we simply move vertex v from I to I^c or vice-versa. Thus the value of $e(I^c)$ changes by at most r since $G \in \mathcal{G}_r$. By applying Theorem 4 to $e(I^c)$, we obtain (2.2). If $p < 1/r$ for r odd and $p < 2/r$ for r even, by (2.1) and (2.2), we have

$$e(I^c) > \left(\left\lceil \frac{r}{2} \right\rceil - 1\right) |I^c| \quad \text{a.a.s.}$$

Lemma 1 with $k = \lceil r/2 \rceil$ implies that the graph $G[I^c]$ induced by I^c a.a.s has a subgraph of minimum degree at least $\lceil r/2 \rceil$, and so I^c a.a.s contains a community. This gives $\theta_p(G) \to 0$, as required.

3 Cubic graphs

In this section, we prove Theorem 2, which determines the dissemination threshold for cubic graphs. We observe that a community in a cubic graph contains a cycle, and therefore the obstruction to a p-insemination overrunning a cubic graph is a cycle of passive vertices.

3.1 Random cubic graphs

In this section, we outline the proof of Theorem 2. To prove that $p_c(\mathcal{G}_3) \le \frac{1}{2}$ we shall find a family of cubic graphs G such that $\theta_p(G) \to 1$ as $|V(G)| \to \infty$ for all $p > \frac{1}{2}$. Note that the existence of such a family is implied by the work of Balogh and Pittel [6]. Nevertheless, our proof is short, self-contained and can be easily turned into an explicit construction of such a family. This family of cubic graphs is generated by considering cubic graphs chosen at random from all cubic graphs, and then showing that such a random graph has the required properties. A survey of random regular graphs is found in Wormald [15]. The specific property we shall require of such graphs G is that the length of the shortest cycle in G tends to infinity as $|V(G)|$ tends to infinity, and G contains no more than 2^i cycles of length i for every $i \le |V(G)|$. We call such graphs *cycle-sparse*. The following fundamental result on short cycles in random regular graphs was proved by Bollobás [3]:

Proposition 1. *Let X_i denote the number of cycles of length i in a random cubic graph on n vertices, for $i \le n$. Then, for any fixed integer $g > 3$,*

$$\lim_{n\to\infty} \mathbb{P}[\forall i \le g : X_i = 0] = \exp\left(- \sum_{i=1}^{g} i^{-1} 2^{i-1} \right).$$

This result was recently extended to longer cycles in random cubic graphs by Garmo [9]. Omitting technical details, the results of Garmo show that for any $i \le n$, $\mathbb{P}[X_i > 2^i] = O(i^{-2})$. Since the Euler sum converges, we deduce that with positive probability $X_i \le 2^i$ for all i. A few more technical considerations show that we can ensure that with positive probability, $X_i = 0$ for $i \le g$ and $X_i \le 2^i$ for $i > g$, no matter what constant value of g we prescribe. It follows that there are infinitely many cycle-sparse cubic graphs.

To finish the proof that $p_c(\mathcal{G}_3) \le \frac{1}{2}$, we fix $p > \frac{1}{2}$ and apply the Harris-Kleitman inequality [2]. For this inequality we consider the probability space $\mathbb{Q}_n$, whose underlying sample space is the n-dimensional Boolean lattice $\{0,1\}^n$ endowed with the natural product probability measure

$$\mathbb{P}(\omega) := \prod_{i=1}^{n} p^{\omega_i}(1-p)^{1-\omega_i} \qquad \text{for } \omega \in \{0,1\}^n.$$

We may consider $\omega \in \{0,1\}^n$ as the incidence vector of a subset of $\{1, 2, \dots, n\}$. Taking this stance, a downset in $\mathbb{Q}_n$ is an event $A \subset \{0,1\}^n$ such that if $\omega \in A$ and $\omega' \subseteq \omega$, then $\omega' \in A$. An event is an upset if its complement is a downset.

Proposition 2. *Let $A_1, A_2, \dots, A_r$ be downsets in $\mathbb{Q}_n$. Then*

$$\mathbb{P}[A_1 \cap A_2 \cap \cdots \cap A_r] \ge \prod_{i=1}^{r} \mathbb{P}[A_i].$$

The same holds if the events are all upsets.

In the current context, we take a p-insemination of a cycle-sparse n-vertex cubic graph G_n (seen as a $\{0,1\}^n$ vector), and observe that the events A_C that all vertices of a cycle $C \subset G_n$ are passive are downsets in Q_n. By the Harris-Kleitman inequality,

$$\mathbb{P}[\bigcap_{C \subset G_n} \overline{A}_C] \;\geq\; \prod_{C \subset G_n} \mathbb{P}[\overline{A}_C]$$

where the products and intersections are over all cycles $C \subset G$. Observe that $\overline{A}_C$ has probability $(1-(1-p)^{\ell})$ if C has length ℓ. Using the cycle-sparse property of G_n, we see

$$\prod_{C \subset G_n} \mathbb{P}[\overline{A}_C] \geq \prod_{i>g} (1-(1-p)^i)^{2^i}.$$

Since $p > \frac{1}{2}$, $1-(1-p)^i > e^{-2(1-p)^i}$. Consequently,

$$\prod_{C \subset G_n} \mathbb{P}[\overline{A}_C] > \exp\Big(2 \sum_{i>g} (2(1-p))^i\Big) > \exp\Big(-\frac{2(2(1-p))^g}{1-2p}\Big).$$

We conclude that for any $p > \frac{1}{2}$ and any constant g,

$$\limsup_{n\to\infty} \theta_p(G_n) \leq 1 - \lim_{n\to\infty} \exp\Big(-\frac{2(2p)^g}{1-2p}\Big).$$

Since g was an arbitrary constant,

$$\lim_{n\to\infty} \theta_p(G_n) = 1$$

and this shows $p_c(\mathcal{G}_3) \leq \frac{1}{2}$.

3.2 $p_c(\mathcal{G}_3) \geq \frac{1}{2}$

With high probability, the existence of many short vertex-disjoint cycles in a cubic graph prevents a p-insemination from overrunning the graph. Therefore, to prove $p_c(\mathcal{G}_3) \geq \frac{1}{2}$, it is enough to consider cubic graphs which have very few short disjoint cycles – after some technical details, we may assume that we have an infinite sequence $\boldsymbol{G}$ where an n-vertex cubic graph G_n in $\boldsymbol{G}$ has no cycles of length at most $2g$ where $g = \frac{1}{8}\log n$. These details will be presented in the full version of the paper. We now outline the proof that for any $p < \frac{1}{2}$ and any increasing sequence of graphs G_n, $\theta_p(G_n) \to 0$ as $n \to \infty$.

Let $C_\lambda(G_n)$ denote the number of sets of λ vertices of G_n through which G_n contains a cycle of length λ – we shall call these cyclic sets. Note that, in general, $C_\lambda(G_n)$ is less than the number of cycles of length λ in G_n. The key idea in showing $\theta_p(G_n) \to 0$ is the following technical proposition:

Proposition 3. *For some λ satisfying $\lambda = \Theta(\log n)$,*

$$C_\lambda(G_n) = \Omega(\lambda^{-4} 2^\lambda).$$

An intuitive way to see this is via eigenvalues: the number of closed walks of length k in G_n is exactly $n\sum_{i=1}^{n}\lambda_i^k$, where λ_i is the ith largest eigenvalue of the adjacency matrix of G_n. Since G_n is cubic, $\lambda_1 = 3$. Now it is possible, although fairly detailed, to show by subtracting walks on trees, that about $\Omega(2^k/k)$ of these walks contain cycles provided k is a large enough constant times $\log n$. A similar computation is carried out in [13] (see Proposition 4.2). Putting $k = \lambda$, and using the girth condition, one arrives at the bound on $C_\lambda(G)$ in Proposition 3. We also observe that in a random cubic graph, the expected number of cycles of length λ is roughly $2^\lambda/\lambda$, so in the sense of counting cycles, G_n is close to a random cubic graph, and these were discussed in the last section. We consider the events A_X that all vertices in a cyclic set X of size λ are passive. The Harris-Kleitman Inequality – Proposition 2 – gives a lower bound on the probability that no A_X occurs, whereas we require an upper bound. The requisite inequality for such an upper bound is Janson's Inequality [16]:

Proposition 4. *Let $A_1, A_2, \ldots, A_r$ be downsets in the probability space $\mathbb{Q}_n$, and define*

$$\triangle = \sum_{i\sim j}\mathbb{P}[A_i \cap A_j]$$

where $i \sim j$ means the events A_i and A_j are dependent and μ is the expected number of A_i which occur. Then

$$\mathbb{P}[\bigcap_{i=1}^{r}\overline{A}_i] \leq e^{-\mu^2/2\triangle}.$$

Showing $\theta_p(G_n) \to 0$ is equivalent to showing that some A_X occurs a.a.s., and we shall establish this with Janson's Inequality by showing that for the events A_X, $\mu^2/\triangle \to \infty$.

To prove this, note that from Proposition 3,

$$\mu = (1-p)^\lambda C_\lambda(G_n) = \Omega\Big(\frac{(2-2p)^\lambda}{\lambda^4}\Big).$$

It is trickier to estimate $\triangle$, and this relies heavily on the assumption that G_n has no cycles of length at most $2g$. To estimate $\triangle$, we fix a cyclic set X and ask, for each $i \in \mathbb{N}$, for the number $\triangle_i(X)$ of cycles C of length λ for which $|X \cap V(C)| = i$. It turns out that

$$\triangle_i(X) = \lambda^{O(1)}2^{\lambda-i-g} \quad \text{for } 1 \leq i < \lambda - g$$

and $\triangle_i(X) = 0$ otherwise. This allows us to estimate $\triangle$:

$$\begin{aligned}\triangle &\leq C_\lambda(G_n) \sum_{i=1}^{\lambda-g-1} (1-p)^{2\lambda-i} \triangle_i(G_n) \\ &= O(\mu^2) \cdot \lambda^{O(1)} \sum_{i=1}^{\lambda-g-1} (1-p)^{-i} 2^{-i-g} \\ &= O(\mu^2) \lambda^{O(1)} 2^{-g}.\end{aligned}$$

Here we used the fact that $p < \frac{1}{2}$. By the choice of g, $\lambda^{O(1)} 2^{-g} \to 0$, and we are done: $\triangle/\mu^2 \to 0$. In words, some λ-cycle is passive a.a.s by Janson's Inequality, and therefore $\theta_p(G_n) \to 0$.

4 Wheels and toroidal grids

We prove here Theorem 3: wheels and toroidal grids plus a universal vertex u have (relatively) small dissemination half-thresholds p_c^+. One of the main observations is that, for any probability $p > 0$, if the universal vertex becomes active during the dissemination process, then the graph is overrun a.a.s. Thus, for any value p such that p-inseminations contaminate a.a.s. more than half of the vertices of the cycle or of the toroidal grid, we deduce that the whole graph is overrun.

There has been much research on dissemination on the k-dimensional torus and grid graphs. The considered rules were the l-neighbours rule, which are more general than the majority rule: in this setting a vertex becomes active if at least l of its neighbours already are active. In particular, Aizenman and Lebowitz [1] studied the 2-neighbours dissemination on P_n^2 and their results extend to C_n^2. Notice that the majority dissemination on C_n^2 is the 3-neighbours dissemination, since C_n^2 is a four-regular graph.

Our approach is based on the observation that once the universal vertex u becomes active, the majority dissemination in the C_n^k part of $u * C_n^k$, in fact, follows the *weak majority* rule restricted to C_n^k. In the weak majority rule a vertex becomes active if at least half of its neighbours are active. If the p-insemination of $u * C_n^k$ is such that half plus one vertex of C_n^k become active, then u becomes active as well. Moreover, for any $p > 0$, the weak majority rule dissemination process for C_n^k will almost surely overrun the whole graph (the result is trivial for cycles, and due to Aizenman and Lebowitz for toroidal grids):

Lemma 2 (see [1]). *Let $O_p^w(G)$ be the random event that a p-insemination overruns G under the* weak majority *rule, and let us denote $o_p^w(G)$ the corresponding probability. Then for any $p > 0$ and any $k \in \{1, 2\}$,*

$$\lim_{n\to\infty} o_p^w(C_n^k) = 1$$

Therefore, for any probability $p > 0$ on graphs of type $u * C_n^k$, if the dissemination contaminates the vertex u it will almost surely overrun the whole graph.

Lemma 3. *Denote by $F_p(G)$ the number of active vertices obtained by the p-dissemination process on G. For every class of graphs $\mathcal{G}$ of type $u * C_n^k$, $p_c^+(\mathcal{G}) = \inf\{p \in [0,1]$ over all values p such that there exists an increasing sequence $u * C_{n_i}^k$ satisfying $\lim_{i\to\infty} \mathbb{P}(F_p(C_{n_i}^k) > n_i^k/2) = 1$.*

From now on we only consider the p-dissemination process in cycles and toroidal grids, under the strong majority rule. Recall that $F_p(G)$ is the random variable counting the number of active vertices in the final state, after a p-dissemination process in G. We give upper and lower bounds for the expected value of F_p for cycles and toroidal grids. Moreover, we shall see that, with very high probability, the value of $F_p(C_n^k)$ is very close to its expectation, when $n \to \infty$. Therefore, it is sufficient to see for which values of p this quantity $\mathbb{E}(F_p(C_n^k))$ is strictly bigger than $n^k/2$, and for which values it is strictly smaller than $n^k/2$. According to Lemma 3, the dissemination threshold for the class $u * C_n^k$ lies between the two values.

Since we are unable to give an exact formula for $F_p(C_n^k)$, we give upper and lower bounds for this quantity. Consider a window $D^d(v)$ formed by all vertices at distance at most d from v in C_n^k. Let $S_p^d(v)$ be a random variable equal to 1 if v becomes active when we replace, in the original p-insemination, all vertices outside the window $D^d(v)$ by passive vertices, and equal to 0 otherwise. Let $s_p^d(C_n^k)$ be the probability that $S_p^d(v) = 1$ (by symmetry this probability is the same for all vertices). Dually, let $W_p^d(v) = 1$ if v becomes active when, in the initial p-insemination, all vertices outside $D^d(v)$ are transformed into active vertices, and $W_p^d(v) = 0$ otherwise. The probability that $W_p^d(v) = 1$ is denoted $w_p^d(C_n^k)$. Finally, let $S_p^d(G) = \sum_v S_p^d(v)$ and $W_p^d(G) = \sum_v W_p^d(v)$[1].

Clearly, we have

Lemma 4. *For any constant d and any $k \geq 1$,*

$$S_p^d(C_n^k) \leq F_p(C_n^k) \leq W_p^d(C_n^k)$$

For any fixed values of k and d, the probabilities $s_p^d(C_n^k)$ and $w_p^d(C_n^k)$ can be expressed as polynomials on p.

Lemma 5.

1. *For any $n \geq 3$,*
$$s_p^1(C_n) = w_p^1(C_n) = p + p^2 - p^3.$$

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[1] In the case of cycles, it is easy to see that the dissemination process stops in exactly one step: a passive vertex becomes active iff both neighbours are active, therefore $S_p^d(C_n) = F_p(C_n) = W_p^d(C_n)$ for any $n \geq 3$ and any $d \geq 1$.

2. *For any $n \geq 5$, $s_p^3(C_n^2)$ and $w_p^3(C_n^2)$ are polynomials of degree 25 on p. Their exact formula has been computed by a program.*

Proof. Let us prove the the first part of the lemma. Let v be a vertex of the cycle and assume that all vertices at distance at least 2 from v are passive. Then v will be active if and only if initially v is already active (which occurs with probability p) or initially v is passive and both his neighbours are active (which occures with probability $(1-p)p^2$. Therefore the probability that u becomes active is $p + p^2 - p^3 = s_p^1$. Now if we configure all non-neighbours of v to be active, the situation is exactly the same: v will be active iff it was active since the begining, or if it was initially passive and both neighbours were active.
For the second part of the proof, the polynomials corresponding to s_p^3 and w_p^3 have been computed by a program. The program considers the window $D^3(v)$ formed by the 25 vertices of distance at most 3 from vertex v in C_n^2. For each number i, with $0 \leq i \leq 25$, we count the number of configurations with exactly i active vertices and such that v belongs to a passive community. (We consider both settings, when vertices outside the window are all active, respectively all passive.) We find e.g. 1 community with 0 active vertices, 24 communities with one active vertex, 276 communities with 2 active vertices, etc. The probability of such a configuration being $p^i(1-p)^{25-i}$, we obtain the required polynomials. □

The expectation of the variable $S_p^d(C_n^k)$ (respectively $W_p^d(C_n^k)$) is $n^k s_p^d(C_n^k)$ (respectively $n^k s_p^d(C_n^k)$). Moreover, we have:

$$S_p^d(C_n^k) \sim n^k s_p^d(C_n^k) \quad \text{and} \quad W_p^d(C_n^k) \sim n^k w_p^d(C_n^k) \quad \text{a.a.s.} \tag{4.1}$$

For proving that the two quantities are very close to their expectations we use again the Independent Bounded Differences inequality (Theorem 4). Consider $S_p^d(C_n^k)$ and $W_p^d(C_n^k)$ as real functions on all possible initial configurations of C_n^k (so their domain is $\{0,1\}^{n^k}$). For each vertex v of C_n^k, let X_v be the random variable s.t. $X_v = 1$ if v is active in the initial configuration, and $X_v = 0$ if v is initially passive. Clearly the variables X_v are independent. Recall that $S_p^d(C_n^k) = \sum_w S_p^d(w)$, where $S_p^d(w)$ is the boolean random variable corresponding to the event "vertex w becomes active if we replace, in the original p-insemination, all vertices at distance larger that d from w by passive vertices". If in the initial configuration we only change the value of one vertex v, this only changes the values $S_p^d(w)$ for vertices w at distance at most d from v. Hence the value of $S_p^d(C_n^k)$ is modified by at most a constant value. By similar arguments, the value of $W_p^d(C_n^k)$ also changes by at most a constant. Therefore we can apply Theorem 4 to both functions, and deduce Equation 4.1.

We are now able to prove our Theorem 3. Consider the case of wheels. For any $p > 0.4030...$, we have $s_p^1(C_n) = p + p^2 - p^3 > 1/2$. By Lemma 4 and Equation 4.1, we have that $F_p(C_n) > n/2$ a.a.s. Therefore $p_c^+(\mathcal{W}) \leq p$, for any $p > 0.4030....$ by Lemma 3. Symmetricaly, for any $p < 0.4030...$, $w_p^1(C_n) < 1/2$

and thus $F_p(C_n) < n/2$ a.a.s. We deduce by Lemma 3 that $p_c^+(\mathcal{W}) \geq 0.4030...$, which proves the first part of Theorem 3.

The same kind of arguments can be applied to toroidal grids plus one vertex. For any $p \geq 0.372$ (resp. any $p \leq 0.35$), the polynomial $s_p^3(C_n^2)$ (resp. $w_p^3(C_n^2)$, see Lemma 5) has value strictly greater (resp. smaller) than 1/2. We conclude by Lemma 3 that $0.35 \leq p_c^+(\mathcal{T}) \leq 0.372$.

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