International Journal of Modern Physics C
Vol. 34, No. 5 (2023) 2350056 (16 pages)

DOI: 10.1142/S0129183123500560

World Scientific
www.worldscientific.com

A large diffusion and small amplification dynamics for density classification on graphs

Laura Leal
Departamento de Ingeniería Matemática
Universidad de Chile, Chile
lleal@dim.uchile.cl

Pedro Montealegre‡
Facultad de Ingeniería y Ciencias
Universidad Adolfo Ibáñez, Chile
p.montealegre@uai.cl

Axel Osses* and Ivan Rapaport†
DIM-CMM (UMI 2807 CNRS)
Universidad de Chile, Chile
**aosses@dim.uchile.cl*
†*rapaport@dim.uchile.cl*

Received 31 May 2022
Revised 16 September 2022
Accepted 21 September 2022
Published 11 November 2022

The density classification problem on graphs consists in finding a local dynamics such that, given a graph and an initial configuration of 0's and 1's assigned to the nodes of the graph, the dynamics converge to the fixed point configuration of all 1's if the fraction of 1's is greater than the critical density (typically 1/2) and, otherwise, it converges to the all 0's fixed point configuration. To solve this problem, we follow the idea proposed in [R. Briceño, P. M. de Espanés, A. Osses and I. Rapaport, Physica D **261**, 70 (2013)], where the authors designed a cellular automaton inspired by two mechanisms: diffusion and amplification. We apply this approach to different well-known graph classes: complete, regular, star, Erdös–Rényi and Barabási–Albert graphs.

Keywords: Automata networks; density classification; Laplacian matrix.

1. Introduction

One of the simplest and regular graph topologies is a torus. A cellular automaton (CA) is nothing but a local dynamics applied to a torus (or to an infinite grid).[2,3] More precisely, a CA consists of n^d cells arranged uniformly in the d-dimensional torus following a local rule identical in every cell. This local rule, which specifies how

‡Corresponding author.

the state of each cell is updated as a function of the states of its neighbor cells, is applied in parallel and in discrete time steps.

In the *density classification problem*, the challenge is to find a CA such that, given any initial configuration x^0 of 0's and 1's, it converges to the all 1's fixed point configuration if the fraction of 1's in x^0 is greater than ρ_c and it converges the all 0's fixed point configuration otherwise. The number $0 < \rho_c < 1$ denotes the *critical density*.

The problem was first formulated for dimension $d = 1$ (a ring) and critical density $\rho_c = 1/2$.[4] The best-known two-state CA for tackling this instance gave a good solution which was not perfect.[5,6] In fact, an impossibility result which says that there is no perfect density classifier with two states was later obtained.[7]

The impossibility of finding perfect classifiers led many researchers to use different evolutionary computation approaches to evolve good *approximate solutions*.[8–12] But in order to obtain perfect density classifiers, researchers were forced to modify the original problem.[13–15] Another idea was to allow the existence of more than one local rule.[16,17] Fatès designed a two-state *stochastic* CA that solves the density classification problem with arbitrary precision.[18]

In Ref. 1, the authors used a continuous approach for solving *deterministically* the density classification problem. More precisely, the idea was to use *local averaging* and *saturation*, a process represented by a bistable heat equation. This bistable model, which exhibits two stable critical points (0 and 1), is a particular case of a reaction–diffusion equation widely used for studying phase transitions and front propagation.[19–21] Similar approaches have been previously used.[22,23] We refer to Ref. 24 for a comprehensive survey about the density classification problem.

The idea of this work is to adapt this approach, which was created for the torus topology, to other topologies corresponding to well-known graph classes: complete, regular, star, Erdös–Rényi and Barabási–Albert graphs.

In this paper, we denote the local dynamics adapted to these graph classes by Φ. Note that Φ corresponds to the discretization of the heat equation and it can in fact be applied to *any* graph.

The idea is the following. Given a critical density $\rho_c \in (0, 1)$, we build a discrete dynamics Φ over an arbitrary connected graph G based on a discrete version of the following equation:

$$\frac{\partial u}{\partial t} - \nu \Delta u = \gamma b_{\rho_c}(u), \tag{1}$$

where $u(x, t)$ is the state at time $t \geq 0$ at point x in a domain Ω. The parameter $\nu > 0$ is a diffusion coefficient, $\gamma > 0$ is an amplification parameter and b_{ρ_c} is a suitable bistable function. In this paper, we choose the cubic polynomial:

$$b_{\rho_c}(u) = u(1 - u)(u - \rho_c).$$

As for the case of the torus, the resulting nonlinear heat equation exhibits two stable critical points (0 and 1, attractors) and one unstable critical point (ρ_c, repulsor).

The main theoretical result of this paper says that, for every connected graph G, there are parameters for the large diffusion and small amplification dynamics Φ, such that Φ solves perfectly the density classification problem in G.

With respect to experimental results, we show how the topology of a given graph G influences the convergence time. First of all, the study of the dynamics *without* amplification — which does not solve the density classification problem — indicates that the convergence time depends on the edge density of G, and that the behavior is fundamentally different on trees (in our simulations, the case of trees is provided by star graphs and a subclass of Barabási–Albert graphs).

Then, we give experimental results when a small amplification is indeed present in Φ. We observe a dramatic difference in the convergence time for the classes that are trees with respect to those that are not (the classes that are not trees, in terms of convergence time, are almost indistinguishable). Finally, we study the influence of the amplification factor on the effectiveness and the convergence time of the local dynamics. Again, we see a dichotomy between trees and graphs with cycles.

2. Preliminaries

In this section, we give the main definitions of the concepts and problems used in the rest of the paper.

We consider only finite simple undirected graphs $G = (V, E)$, where V is the set of vertices and $E \subseteq \binom{V}{2}$ is the set of (undirected) edges. For a node v in a graph $G = (V, E)$, we call $N(v) = \{u \in V : \{u, v\} \in E\}$ the neighborhood of v. The cardinality of $N(v)$ is called the *degree* of v and it is denoted as $d(v)$. The *maximum degree* is denoted as $\Delta(G)$ and corresponds to $\max_{v \in V} d(v)$. Given a graph G with $V(G) = \{v_1, \ldots, v_n\}$, the *adjacency matrixA* of G corresponds to a square matrix of order n such that $A_{ij} = 1$ when $\{v_i, v_j\} \in E(G)$, and $A_{ij} = 0$ otherwise.

A sequence of vertices $P = v_1, \ldots, v_k$ is called a v_1, v_k-*path* if $\{v_i, v_{i+1}\}$ is an edge of G, for each $1 \leq i < k$. Two vertices u and v are *connected* if there exists a u, v-path in G. Being connected defines an *equivalence relation*, and the equivalence classes of this relation (i.e. inclusion maximal sets of connected vertices) are called *connected components* of G. In this paper, we consider only connected graphs. A *cycle* is a sequence of $k \geq 3$ vertices $v_1, \ldots, v_k$ that form a v_1, v_k-path and where $\{v_1, v_k\}$ is an edge of G. A connected graph without cycles is called a *tree*.

We denote by $[n]$ the set $\{1, \ldots, n\}$. Given a graph $G = ([n], E)$, a *configuration* is a vector $u \in [0, 1]^n$. In this paper, we study time-discrete dynamical systems over the configurations of graphs. Formally, we study the dynamics given by certain *local updating rules*, which are functions $F : [0, 1]^n \to [0, 1]^n$. Such a function F defines a sequence of configurations given by

$$u^t = F(u^{t-1}),$$

for each $t \geq 1$, where $u^0 \in [0, 1]^n$ is denoted as the *initial configuration*.

Given a configuration $u \in [0,1]^n$, the *density of* u, denoted by $\rho(u)$, is the quantity

$$\rho(u) = \sum_{i \in [n]} \frac{u_i}{n}.$$

In that context, given a *critical density* $\rho_c \in (0,1)$, the *density classification problem* consists of the design of a time-discrete dynamical system F such that, for every Boolean initial configuration $u \in \{0,1\}^n$:

- If $\rho(u) < \rho_c$ then $\lim_{t\to\infty} F^t = [0,\ldots,0]$.
- If $\rho(u) > \rho_c$ then $\lim_{t\to\infty} F^t = [1,\ldots,1]$.

In other words, a rule F solves the density classification problem if, for every Boolean initial configuration, the dynamics converge to either an all 1's or all 0's configuration, depending on whether the density of the initial configuration is below or above a given critical density.

We also consider an approximated version of the density classification problem. Given $\varepsilon > 0$, we say a rule F solves the ε-approximation of the density classification problem if

- If $\rho(u) < \rho_c - \varepsilon$ then $\lim_{t\to\infty} F^t = [0,\ldots,0]$.
- If $\rho(u) > \rho_c + \varepsilon$ then $\lim_{t\to\infty} F^t = [1,\ldots,1]$.

3. A Local Dynamics for Density Classification

In this section, we define the large diffusion and small amplification dynamics. They are defined by a local rule Φ which is based on the discretization of a bistable nonlinear heat equation. More precisely, given a critical density $\rho_c \in (0,1)$, the idea is to build a discrete dynamic over an arbitrary connected graph G based on a discrete version of the following equation:

$$\frac{\partial u}{\partial t} - \nu \Delta u = \gamma b_{\rho_c}(u), \tag{2}$$

where $u(x,t)$ is the state at time $t \geq 0$ at point x in a domain Ω. The parameter $\nu > 0$ is a diffusion coefficient, $\gamma > 0$ is an amplification parameter and b_{ρ_c} is a suitable bistable function. In this paper, we choose the cubic polynomial

$$b_{\rho_c}(u) = u(1-u)(u-\rho_c).$$

The resulting nonlinear heat equation is called the bistable heat equation, since it exhibits two stable critical points (0 and 1, attractors) and one unstable critical point $(\rho_c, \text{repulsor})$.

Let $G = (V, E)$ be a graph with vertex set $V = \{1,\ldots,n\}$ and edge set $E \subseteq \binom{V}{2}$. We denote by $N(i)$ the set of neighbors of node i, and by d_i the degree (number of neighbors) of i. We can discretize Eq. (1) with an explicit finite differences scheme on a uniform space difference $h > 0$ and discrete time steps $t_k = k\Delta t$ for some $\Delta t > 0$,

obtaining

$$\frac{u_i^{k+1} - u_i^k}{\Delta t} + \frac{\nu}{h^2} \sum_{j \in N(i)} L_{ij} u_j^k = \gamma b_{\rho_c}(u_i^k),$$

where $L = L(G)$ is the Laplacian matrix of G, which corresponds to an $n \times n$ matrix defined by

$$L_{ij} := \begin{cases} d(i) & \text{si } i = j, \\ -1 & \text{si } ij \in E, \\ 0 & \text{si } ij \notin E. \end{cases}$$

If we denote by $\sigma = \Delta t \gamma$, $\sigma' = \frac{\nu \Delta t}{h^2}$ and $b_{\rho_c}(u_i) = u_i(1 - u_i)(u_i - \rho_c)$, we obtain the local rule

$$u_i^{k+1} = u_i^k - \sigma' \sum_{j \in N(i)} L_{ij} u_j^k + \sigma b_{\rho_c}(u_i^k).$$

Finally, we choose the parameter ν in order to fix $\sigma' = \frac{1}{(\Delta+1)}$, where $\Delta = \Delta(G)$ is the maximum degree of G. Then, the *large diffusion and small amplification dynamic over* G updating rule Φ_σ

$$v^t = \left(I - \frac{L}{\Delta + 1} \right) u^t, \tag{3}$$

$$u_i^{t+1} = f_\sigma(v_i^t), \tag{4}$$

where $f_\sigma(x) = x + \sigma b_{\rho_c}(x)$.

Theorem 1. *For every connected graph G and for every $\varepsilon > 0$ there exists a $\sigma > 0$ such that the large diffusion and small amplification dynamic over G solves the ε-approximation of the density classification problem.*

The proof of Theorem 1 is analogous to the proofs given in Ref. 1. For sake of completeness we include all the details. In order to prove Theorem 1, we prove first some technical lemmas. In the first lemma, we show that the dynamic without amplification (i.e. when $\sigma = 0$) converges to a configuration where the state of every vertex is the density of the initial configuration.

In the following, we denote by C the matrix $(I - \frac{L}{\Delta+1})$.

Lemma 1. *For every $u \in \{0,1\}^n$, $\lim_{t \to \infty} \Phi_0^t(u) = \rho(u)[1, \ldots, 1]^T$.*

Proof. When we consider the dynamics without any amplification $\sigma = 0$, we obtain that for every $t > 0$, the updating rule becomes

$$u^t = \Phi_0(u^t) = C u^{t-1} = C^t u^0.$$

Observe that C is a doubly stochastic (its entries are not nonnegative and the sum of their rows and columns is 1) and symmetric matrix. Moreover, C is primitive, meaning that there exists an $m > 0$ such that $(C^m)_{ij} \neq 0$ for all i, j. The existence of

such m simply follows from the fact that the underlying graph is connected. Because C is symmetric, we know that all of its eigenvalues are real.

From the properties of C and the Perron–Frobenius theorem, we deduce the following properties. First, $\lambda = 1$ is an eigenvalue of C of multiplicity one. Also, the eigenspace associated with the eigenvalue $\lambda = 1$ is spanned by $[1 \cdots 1]^T$. The absolute value of all the other eigenvalues is strictly less than 1. We obtain that C can be decomposed as $M^T D M$, where D is the diagonal, the elements on the main diagonal of D are the eigenvalues of C and M that are orthonormal. Moreover,

$$\lim_{t\to\infty} D^t = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

implying that

$$\lim_{t\to\infty} C^t = \frac{U}{n},$$

where U is the square matrix of order n where all its entries are ones. We deduce that for every initial configuration $u^0 \in \{0,1\}^n$,

$$\lim_{t\to\infty} u^{t+1} = \lim_{t\to\infty} C^t u^0 = \frac{1}{n}\left(\sum_{i=1}^{n} u_i^0\right)[1 \cdots 1]^T = \rho(u^0)[1 \cdots 1]^T.$$

Implying that when $\sigma = 0$ we have that the dynamics converge to a configuration where the state of every node corresponds to the density $\rho(u^0)$ of the initial configuration u^0. □

Let us now study the case when $\sigma > 0$. When $\sigma > 0$ we have that f_σ is strictly increasing, $f_\sigma(0) = 0$ and $f_\sigma(1) = 1$. Therefore, by continuity f_σ is a one-to-one map from $[0,1]$ to $[0,1]$. In the following lemma, we show that when the initial configuration has each coordinate greater than (respectively, smaller than) the critical density ρ_c, then the dynamic correctly solves the density classification problem.

Lemma 2. *Let $\sigma > 0$. If $u \in [0,1]^n$ is such that, for every $i \in [n]$, $u_i > \rho_c$ then,*

$$\lim_{t\to\infty} \Phi_\sigma^t(u) = [1, \ldots, 1]^n.$$

Similarly, if $u \in [0,1]^n$ is such that, for every $i \in [n]$, $u_i < \rho_c$ then,

$$\lim_{t\to\infty} \Phi_\sigma^t(u) = [0, \ldots, 0]^n.$$

Proof. Let us consider first a uniform initial configuration $u^0 = [q, \ldots, q]^T$ with $q \in (\rho_c, 1]$. Then $v^0 = Cu^0 = [q, \ldots, q]^T = u^0$. Since f_σ is increasing, we obtain that for every $i \in [n]$, $u_i^0 < f_\sigma(u_i^0) = \Phi_\sigma(u)_i \leq 1$. Inductively, for every $t \geq 1$ and every $i \in [n]$, $\Phi_\sigma^{t-1}(u)_i < \Phi_\sigma^t(u)_i \leq 1$, we deduce that $\lim_{t\to\infty} \Phi_\sigma^t(u) = [1, \ldots, 1]^T$.

Now let us fix a configuration $u \in [0,1]^n$ such that $\rho_c < u_i$ for all $i \in [n]$, and let us call $q = \min_{i\in[n]} u_i \in (\rho_c, 1]$. Then $v = Cu$ satisfies that $\rho_c < q \leq v_i$ for every $i \in [n]$. Therefore $\rho_c < q \leq v_i \leq f_\sigma(v_i)$. This implies that $\Phi(u)$ is a configuration in $[0,1]^n$ such that $\rho_c < \Phi(u)_i$ for all $i \in [n]$, and such that $q \leq \min_{i\in[n]} \Phi(u)_i$.

Now define $u^* = [q, \ldots, q]^T$. From previous remarks, we have that $\lim_{t\to\infty} \Phi^t(u^*) = [1 \cdots 1]^T$ and that for every $t \geq 0$ and $i \in [n]$, $\Phi^t(u^*)_i \leq \Phi^t(u)_i \leq 1$, we deduce that $\lim_{t\to\infty} \Phi^t(u) = [1 \cdots 1]^T$.

The case when $\rho_c > u_i$ for all $i \in [n]$ is analogous. □

In the following lemma, we bound the difference between the dynamic with amplification respect to the dynamic without amplification. Before giving the lemma, consider the following two remarks. First observe that, if $x \in [0,1]$,

$$-\frac{\rho_c}{4} \leq b_{\rho_c}(x) \leq \frac{(1-\rho_c)}{4}.$$

Indeed, as $x(1-x)$ reaches its maximum at $x = 1/2$. Then, we have that

$$-\frac{\rho_c}{4} \leq -\rho_c x(1-x) \leq b_{\rho_c}(x) \leq (1-\rho_c)x(1-x) \leq \frac{1-\rho_c}{4}.$$

Second, notice that since all the coordinates of matrix C are positive, then Φ_0 is monotone. More precisely, for every pair of configurations $w^1, w^2 \in [0,1]^n$, such that for all $i \in [n]$ $w_i^1 \leq w_i^2$, we have that $(Cw^1)_i \leq (Cw^2)_i$.

Lemma 3. *Let $u \in [0,1]^n$ be an initial configuration and $\sigma > 0$. Then, for every $t \geq 0$ and $i \in [n]$,*

$$\min_{j\in[n]} (C^t u)_j - \frac{\sigma t \rho_c}{4} \leq \Phi_\sigma^t(u)_i \leq \max_{j\in[n]} (C^t u)_j - \frac{\sigma t(1-\rho_c)}{4}.$$

Proof. Let $u \in [0,1]^n$ and $\sigma > 0$. We now show by induction on $t \geq 1$ that

$$(C^t u)_i - \frac{\sigma t \rho_c}{4} \leq \Phi_\sigma^t(u).$$

In the base case $t = 1$, we have that

$$\begin{aligned}\Phi_\sigma(u)_i &= (Cu)_i + \sigma b_{\rho_c}((Cu)_i) \\ &\geq Cu_i - \frac{\rho_c \sigma}{4}.\end{aligned}$$

Now, for the inductive step, suppose that the property holds for $t \geq 1$. Then

$$\begin{aligned}\Phi_\sigma^{t+1}(u)_i &= (C\Phi_\sigma^t(u))_i + \sigma b_{\rho_c}((C\Phi_\sigma^t(u)_i) \\ &\geq (C\Phi_\sigma^t(u))_i - \frac{\rho_c \sigma}{4}.\end{aligned}$$

By the induction hypothesis, we know that $\Phi_\sigma^t(u) \geq C^t u - \frac{\rho_c t\sigma}{4}[1,\ldots,1]^T$. Then, the second remark above implies that

$$\begin{aligned}\Phi_\sigma^{t+1}(u)_i &\geq \left(C\left(C^t u - \frac{\rho_c t\sigma}{4}[1,\ldots,1]^T\right)\right)_i - \frac{\rho_c\sigma}{4}\\ &\geq \left(C^{t+1}u - \frac{\rho_c t\sigma}{4}C[1,\ldots,1]^T\right)\right)_i - \frac{\rho_c\sigma}{4}\\ &\geq (C^{t+1}u)_i - \frac{\rho_c t\sigma}{4} - \frac{\rho_c\sigma}{4}\\ &\geq (C^{t+1}u)_i - \frac{\rho_c(t+1)\sigma}{4}.\end{aligned}$$

We deduce that

$$\Phi_\sigma^t(u) \geq (C^t u)_i - \frac{\sigma t\rho_c}{4} \geq \min_{j\in[n]}(Cu)_j - \frac{\sigma t\rho_c}{4}.$$

By analogous arguments, we conclude that

$$\Phi_\sigma^t(u)_i \leq \max_{j\in[n]}(C^t u)_j - \frac{\sigma t(1-\rho_c)}{4}.$$

We are now ready to give the proof of Theorem 1. □

Proof of Theorem 1. Without loss of generality, let us assume that $\rho_c \geq 1/2$. Since in a finite dimension, all norms are equivalent, we have from Lemma 1 that when $t \to \infty$:

$$\max_{u\in[0,1]^n}\left\|\left(C^t - \frac{U}{n}\right)u\right\|_\infty \to 0$$

this implies that there exists a $t_0 = t_0(n,\varepsilon)$ such that for every $t \geq t_0$,

$$\max_{u\in[0,1]^n}\left\|\left(C^t - \frac{U}{n}\right)u\right\|_\infty \leq \frac{\varepsilon}{3}.$$

Then, from Lemma 3 applied to t_0 we have that for all $i \in [n]$

$$\min_{j\in[n]}(C^{t_0}u)_j - \frac{\sigma t_0\rho_c}{4} \leq \Phi_\sigma^{t_0}(u)_i \leq \max_{j\in[n]}(C^{t_0}u)_j - \frac{\sigma t_0(1-\rho_c)}{4}.$$

Hence,

$$\rho(u) - \frac{\varepsilon}{3} - \frac{\sigma t_0\rho_c}{4} \leq \Phi_\sigma^{t_0}(u)_i \leq \rho(u) + \frac{\varepsilon}{3} - \frac{\sigma t_0(1-\rho_c)}{4}.$$

Thus, for every $i \in [n]$,

$$|\rho(u) - \Phi_\sigma^{t_0}(u)_i| \leq \frac{\varepsilon}{3} + \frac{\sigma t_0\rho_c}{4}.$$

Now, let us pick $\sigma = \frac{4\varepsilon}{3t_0\rho_c}$ in order to obtain

$$|\rho(u) - \Phi_\sigma^{t_0}(u)_i| \leq \frac{2\varepsilon}{3}.$$

If u is such that $\rho(u) > \rho_c + \varepsilon$, then $\Phi^{t_0}(u)_i > \rho_c$ for every $i \in [n]$. In other words, after t_0 iterations of the dynamic, all the coordinates of the reached configuration has all its coordinates greater than ρ_c. From Lemma 2, we deduce that

$$\lim_{t\to\infty} \Phi_\sigma^t(u) = [1, \ldots, 1]^T.$$

By analogous arguments, when $\rho(u) < \rho_c - \varepsilon$ we deduce that

$$\lim_{t\to\infty} \Phi_\sigma^t(u) = [0, \ldots, 0]^T.$$

□

4. Experimental Results

In this section, we report the results of an empirical study of the large diffusion and small amplification dynamics. Our goal is to describe how the topology of a given graph G influences the convergence time and the effectiveness of the classification for fixed values of σ (we are considering the situation where the critical density $\rho_c = 1$). We evaluate the dynamics on topologies having different densities (total number of edges), and different degree distributions.

Our study is carried out considering the following graph classes of connected graphs:

- **Complete graphs** K_n, which correspond to the n-node graphs with all possible edges.
- **Regular graphs**, which are the n-node graphs where each node has the same degree d. In our experiments, we consider $d \in \{4, 6, 8\}$.
- **Star graphs** S_n, consisting of the n-node graphs where one node has degree $n-1$ and the other $n-1$ nodes have degree 1.
- **Erdös–Rény graphs**,[25] which is a model of random graphs, where each edge is independently included in the graph with probability p. For our experiments, for an n-node graph, we pick $p = \frac{2\ln(n)}{n}$, which roughly corresponds to the minimum probability that ensures that the graph is connected with high probability.[25]
- **Barabási–Albert graphs**,[26] which is another model of random graphs, is generated sequentially according to a parameter m as follows. The network starts with m nodes connected randomly. Then, $n-m$ nodes are sequentially added to the graph. Each time a new node is added, it is connected to m existing nodes with a probability that is proportional to the degree that existing nodes already have. More formally, for each $i \in \{m+1, \ldots, n\}$, let us call $|E(G_i)|$ the total number of edges in the graph induced by $G_i = G[\{1, \ldots, i-1\}]$ and $d_i(v)$ the degree of node $v \in \{1, \ldots, i-1\}$ on G_i. Then, when node i is included in the graph, it is connected

to m random neighbors, where node $v \in G_i$ is picked with probability $\frac{d_i(v)}{|E(G_i)|}$. Observe that Barabási–Albert graphs with parameter m are trees.

The choice of the graph classes is based on the following criteria. First, complete graphs are used as a benchmark of ideally connected topologies, for which the local dynamics behaves in the most efficient way.

Second, the dynamics over regular graphs corresponds to the most natural generalization of the dynamics given in Ref. 1, which is designed specifically for d-dimensional tori, which are a particular case of $2d$-regular graphs.

Third, star graphs correspond to the extreme opposite of regular graphs, where one node has maximum degree, and all the others have minimum degree.

Finally, Erdös–Rényi and Barabási–Albert graphs are the best-known models of random graphs, which are commonly used to represent real-world graphs such as social networks, biological systems or particle systems.

For our simulations, we pick odd values of n, from 11 to 251 with a step of 30. For each n, we pick the n-node complete graph K_n, the n-node star S_n, and, also, 50 independent random samples of

- d-Regular graphs for $d \in \{4, 6, 8\}$ (50 samples for each d);
- Erdös–Rényi graphs with parameter $p = \frac{(2\ln(n))}{n}$;
- Barabási–Albert graphs of parameter $m \in \{1, 2, 3, 4\}$ (50 samples for each m).

If a sampled graph is not connected, it is discarded and not considered in the statistics. For each n and for each graph, $10 \cdot n$ random initial configurations are picked uniformly at random.

4.1. *Experimental study when* $\sigma = 0$

We first study the convergence time of the dynamics without amplification (i.e. when $\sigma = 0$). The dynamics is iterated until the difference between each pair of coordinates of two consecutive configurations is smaller than 10^{-3}. Finally, we take the mean convergence time over all configurations and all graphs sampled in the corresponding class.

Our empiric study of the dynamic without amplification indicates that the convergence time is affected by the edge density of the input graph, and that it is fundamentally different on trees. The results are reported in Fig. 1.

Our results show that for complete graphs, Erdös–Rényi graphs and regular graphs, the convergence time exhibits a logarithmic growth with respect to the size of the graph. Obviously, complete graphs have the lowest mean convergence time. In the case of regular graphs, the convergence time exhibits a growth inversely proportional to the degree.

Interestingly, Erdös–Rényi graphs have the greatest mean convergence time of this group, despite the fact that that each node has an expected degree $2\ln(n)$. Also, for these graphs, the convergence time exhibits a greater standard deviation,

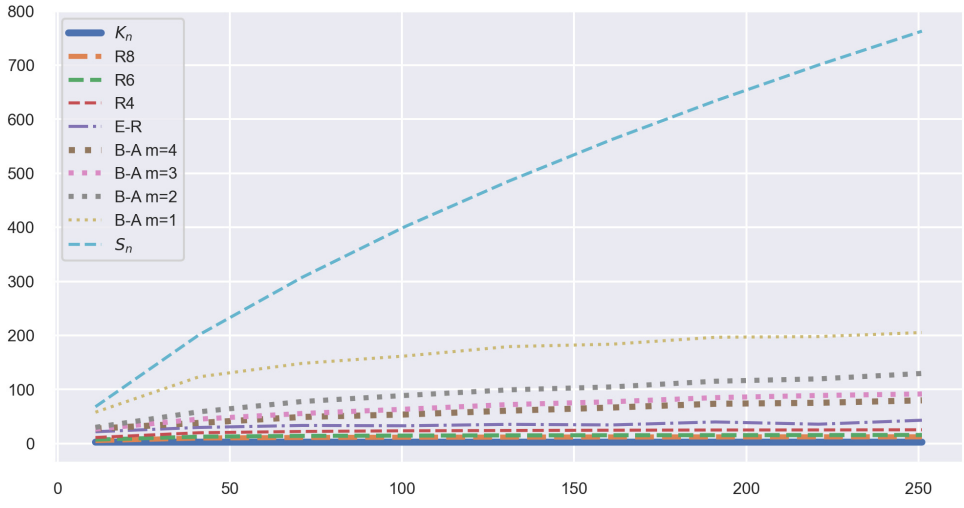

Fig. 1. Mean convergence time of the dynamics without amplification ($\sigma = 0$). Each line represents the mean convergence time of the dynamic in the corresponding set of samples. Lines labeled K_n and S_n represent the complete and star graphs. Lines labeled R4, R6, R8 denote mean convergence time of samples of d-regular graphs for d equals 4, 6 and 8, respectively. The lines labeled E-R represent the convergence time of the samples of the Erdös–Rényi graphs. Finally, lines $B - Am = k$ represent the convergence time of the samples of the Barabási–Albert of parameters $k \in \{1, 2, 3, 4\}$.

implying that the curve is less smooth than the other ones. Nevertheless, the mean convergence time of Erdös–Rényi graphs still follows a logarithmic growth.

A second group is formed by the Barabási–Albert graphs with parameter m different than 1. For this group, the convergence time exhibits a linear growth, and it is inversely proportional to m.

Finally, a third group is formed by the star graphs and the Barabási–Albert graphs with parameter $m = 1$. Observe that these two classes are subclasses of trees. These classes exhibit a mean convergence time that is much larger than the others.

4.2. *Experimental study when* $\sigma = 0.01$ *and* $\rho_c = \frac{1}{2}$

In this section, we report the experimental results when a small amplification is introduced. For simplicity, our study is carried out for a critical density $\rho_c = \frac{1}{2}$. We empirically tested different values of σ in order to obtain a 100% correct classification on every configuration and graph we tested. A report over the effectiveness vs correctness can be found in the following section.

We fix $\sigma = 0.01$ and apply the same simulation framework that we applied in the previous section. However, in this case, the convergence time is approximated by the time-step on which either every node has a state greater than 0.99 (i.e. when an all 1's configuration is approximately reached), or every node has a state smaller than 0.01 (i.e. when an all 0's configuration is approximately reached). In the first case, we assume that the dynamics has classified the initial configuration with a density

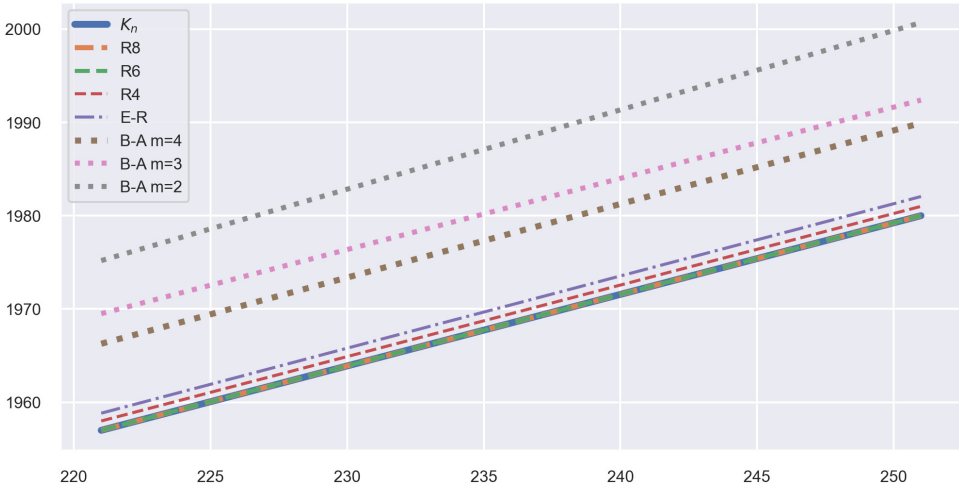

Fig. 2. Mean convergence time of the dynamics with amplification $\sigma = 0.01$ for graph sizes between 221 and 251. Each line represents the mean convergence time of the dynamics in the corresponding set of samples. Line labeled K_n represents the complete graph. Lines labeled R4, R6, R8 denote mean convergence time of samples of d-regular graphs for d equals 4, 6 and 8, respectively. The lines labeled E-R represent the convergence time of the samples of the Erdös–Rényi graphs. Finally lines B-A $m = k$ represent the convergence time of the samples of the Barabási–Albert with parameters $k \in \{2, 3, 4\}$.

greater than $\frac{1}{2}$, while in the second case, we assume that the dynamics has classified the initial configuration with a density smaller than $\frac{1}{2}$.

From our experimental results, we deduce the following observations.

First, we obtain a dramatic difference on the convergence time for the classes that are not trees, namely, complete, regular, Erdös–Rényi and Barabási–Albert graphs with parameter m different than 1. For all these graph classes, the convergence time is almost indistinguishable. In fact, the differences are roughly the same that we observed in the convergence time of the dynamics without amplification. For clarity, we show in Fig. 2 only the convergence times for graph sizes in the rank between 221 and 251.

Second, for the graph classes that are trees, namely, the star graph and the Barabási–Albert graphs with parameter $m = 1$, we observed that the convergence time diverges from the one observed for the other graph classes. This difference is especially dramatic for the Barabási–Albert graphs with parameter $m = 1$ (see Fig. 3).

4.3. *Convergence time and effectiveness vs amplification*

In this last section, we study the influence of the amplification factor on the effectiveness and the convergence time of the dynamics. For $n = 11$ and values of σ from 0.01 to 1, we compute the number of iterations on which the dynamic reaches an (approximately) all 1's configuration or an (approximately) all 0's configuration, with the same criteria of last section. The results are reported in Figs. 4 and 5.

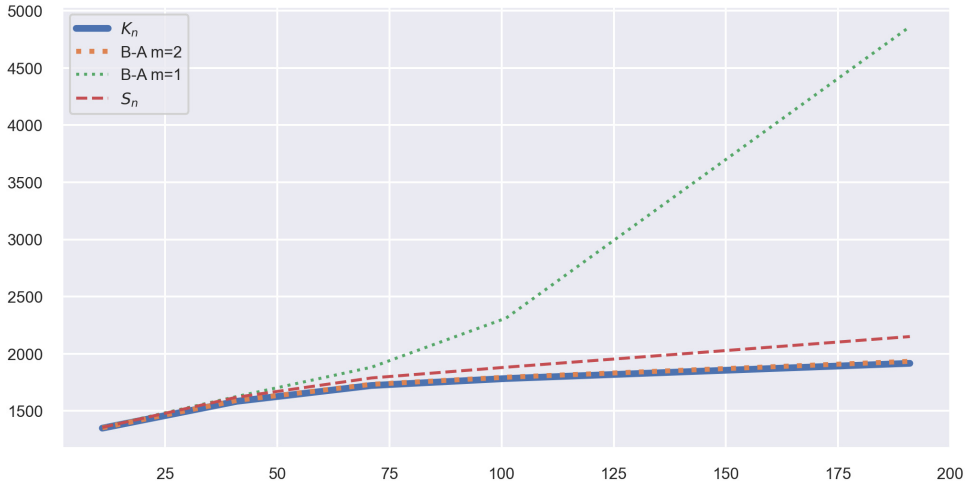

Fig. 3. Mean convergence time of the dynamics with amplification $\sigma = 0.01$ for different graph sizes. Each line represents the mean convergence time of the dynamics in the corresponding set of samples. Lines labeled K_n and S_n represent the complete and the star graphs. Lines B-A $m = k$ represent the convergence time of the samples of the Barabási–Albert of parameters $k \in \{1, 2\}$.

The results for complete, regular, Erdös–Rényi and Barabási–Albert graphs with parameter m different than 1 are very similar. In all these classes, the behavior is roughly the same. For that reason, in Fig. 4, we choose the complete graph as a representative of this set of graph classes. For $\sigma > 0.1$, the convergence time is roughly small (bounded by 200 time-steps), while for $\sigma < 0.1$, the convergence time

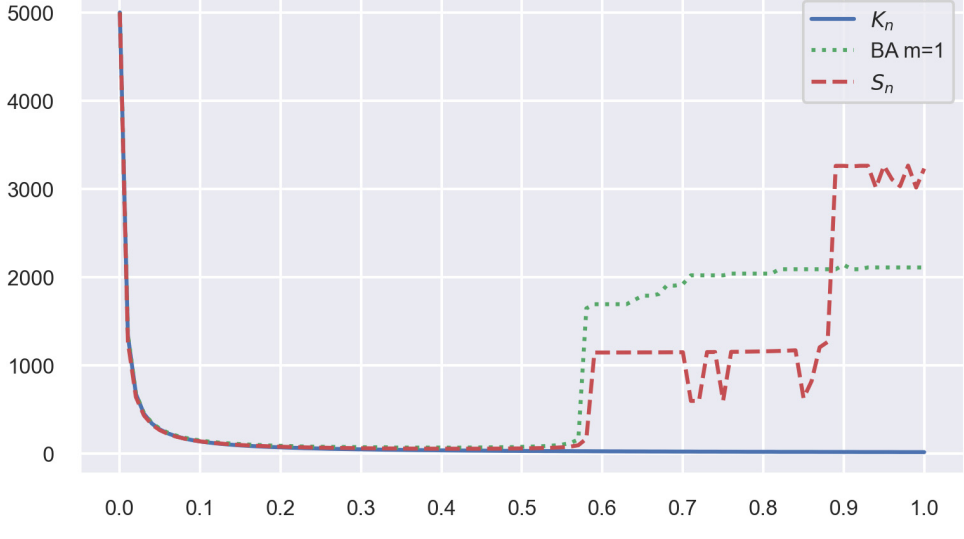

Fig. 4. Mean convergence time for different values of σ for the 11 node complete graphs, star graphs and Barabási–Albert graphs with parameter $m = 1$.

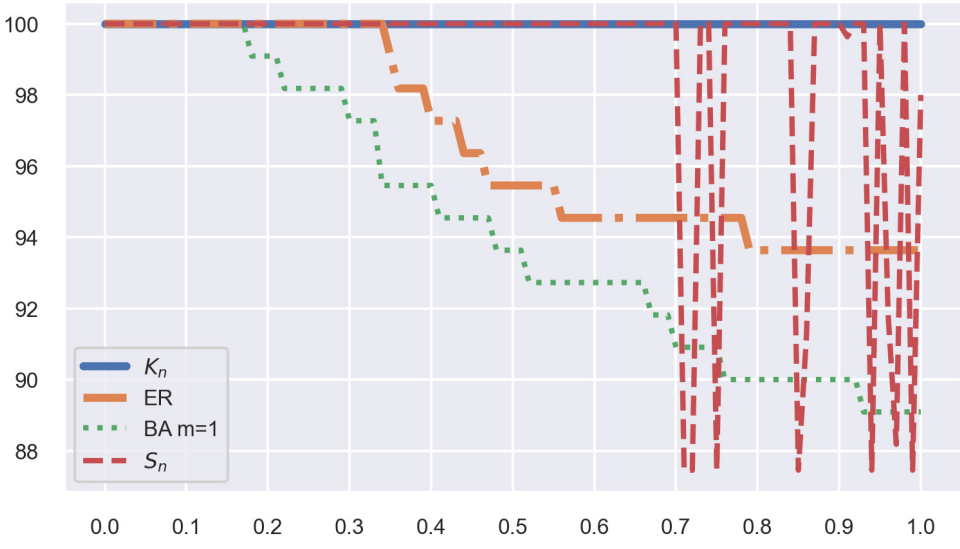

Fig. 5. Mean percentage of effectiveness for different values of σ for the 11 node complete, Erdös–Rényi, star and Barabási–Albert graphs with parameter $m = 1$.

grows exponentially. For the complete, regular and Barabási–Albert graphs with parameter $m \in \{3, 4\}$, the classification effectiveness was a 100% on every $\sigma \leq 1$. On the contrary, for the Erdös–Rényi graphs and the Barabási–Albert graphs with parameter $m = 2$, the classification effectiveness drops from 100% for $\sigma > 0.35$ (see Fig. 5).

As we may expect from previous simulations, the behavior differs when we analyze the star graphs or the Barabási–Albert graphs with parameter $m = 1$. As for the other classes, the convergence time diverges when σ is smaller than 0.1. However, in the case of the star graph, we found that the effectiveness of classification drops from 100% when $\sigma > 0.6$. Around that threshold (for $\sigma > 0.56$), the convergence time tends to augment, but it does not explode as for small values of σ. Interestingly, the effectiveness star graphs oscillate between an 88% and a 100% as σ approaches 1. Finally, with respect to Barabási–Albert graphs with parameter $m = 1$, we observe a behavior that is similar to the star graphs, with two relevant differences. First, the classification effectiveness drops from 100% when $\sigma > 0.18$, which is smaller than the observed threshold for star graphs. Second, as σ approaches 1 the drop in the effectiveness is monotonic.

5. Conclusion

In its original formulation, the density classification problem is stated for one-dimensional cellular automata. Since this problem has no solution, various efforts have been carried out in order to try to solve it perfectly in other formulations. In Ref. 1, the problem was stated in terms of the binary heat equation, with continuous

states between 0 and 1, and shown that it could be solved on any d-dimensional regular grid with a sufficiently small value of the parameter σ, denoted as *amplification factor*. In this paper, we have proposed a generalization of this formulation to arbitrary graphs.

From the theoretical point of view, we have shown that any connected graph admits a solution for a sufficiently small amplification parameter. Later, we reported an experimental study of the dynamics. Our goal was to describe how the topology of a given graph G influences the convergence time and the effectiveness of the classification for fixed values of σ. Roughly, our results indicate that there is a noticeable difference between dense and sparse topologies with respect to the size of the amplification factor. In particular, connected acyclic graphs (trees) such as star graphs and Barabási–Albert graphs with parameter 1, require a smaller amplification factor compared to more dense topologies, such as Erdös–Rényi or regular graphs, in order to obtain comparable classification effectiveness. Since the convergence time of the dynamics is indirectly proportional to the size of σ, our results suggest that, from the computational complexity point of view, the density classification problem may be harder on sparse topologies than on dense topologies. The theoretical verification of this claim is a matter of future research.

Acknowledgments

This research was possible by the support of the supercomputing infrastructure of the NLHPC (ECM-02) and the support of Centro de Modelamiento Matemático (CMM), FB210005 BASAL funds for centers of excellence from ANID-Chile. L. L. acknowledges the Chilean government through the National Agency for Research and Development (ANID)/Scholarship Program/BECA DOCTORADO NACIONAL/2019-21191440. P. M. acknowledges ANID-FONDECYT 11190482 and ANDO-PAI 77170068. A. O. acknowledges ANID-Fondecyt 1191903, 1201311, FONDAP/15110009, Millennium Program NCN19-161, ACIPDE MATH190008 and Climat-Amsud project CLI2020008. I. R. acknowledges ANID-FONDECYT 1220142.

References

1. R. Briceño, P. M. de Espanés, A. Osses and I. Rapaport, *Physica D* **261**, 70 (2013).
2. J. Von Neumann *et al.*, *IEEE Trans. Neural Netw.* **5**, 3 (1966).
3. S. Ulam, Random processes and transformations, in *Proc. Int. Congress on Mathematics* (30 August-6 September 1950), pp. 264–275.
4. N. H. Packard, *Dyn. Patterns Complex Syst.* **212**, 293 (1988).
5. P. Gács, G. L. Kurdyumov and L. A. Levin, *Probl. Peredachi Inf.* **14**, 92 (1978).
6. P. G. de Sá and C. Maes, *J. Stat. Phys.* **67**, 507 (1992).
7. M. Land and R. K. Belew, *Phys. Rev. Lett.* **74**, 5148 (1995).
8. H. Juille and J. B. Pollack, Coevolving the "ideal" trainer: Application to the discovery of cellular automata rules, in University of Wisconsin (1998).
9. M. Mitchell, J. P. Crutchfield and P. T. Hraber, *Physica D* **75**, 361 (1994).

10. P. P. de Oliveira, J. C. Bortot and G. M. Oliveira, *Neurocomputing* **70**, 35 (2006).
11. S. Verel, P. Collard, M. Tomassini and L. Vanneschi, *Theor. Comput. Sci.* **378**, 54 (2007).
12. D. Wolz and P. P. De Oliveira, *J. Cell. Autom.* **3**, 289 (2008).
13. M. S. Capcarrere, M. Sipper and M. Tomassini, *Phys. Rev. Lett.* **77**, 4969 (1996).
14. M. Dembowski, B. Wolnik, W. Bołt, J. M. Baetens and B. De Baets, *J. Cell. Autom.* **14** (2019).
15. B. Wolnik, M. Dembowski, W. Bołt, J. M. Baetens and B. D. Baets, The density classification problem in the context of continuous cellular automata, in *Cellular Automata. ACRI 2016*, S. El Yacoubi, J. Wąs, S. Bandini (eds.), Lecture Notes in Computer Science, Vol. 9863 (Springer, Cham, 2016), https://doi.org/10.1007/978-3-319-44365-2_8.
16. H. Fukś, *Phys. Rev. E* **55**, R2081 (1997).
17. C. L. Martins and P. P. de Oliveira, Evolving sequential combinations of elementary cellular automata rules, in *Advances in Artificial Life. ECAL 2005*, M. S. Capcarrère, A. A. Freitas, P. J. Bentley, C. G. Johnson and J. Timmis (eds.), Lecture Notes in Computer Science, Vol. 3630 (Springer, Berlin, Heidelberg, 2005), https://doi.org/10.1007/11553090_47.
18. N. Fatès, Stochastic cellular automata solve the density classification problem with an arbitrary precision, in *28th International Symposium on Theoretical Aspects of Computer Science (STACS 2011)*, Thomas Schwentick and Christoph Dürr (eds.) (Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2011), http://drops.dagstuhl.de/opus/volltexte/2011/3018.
19. R. S. Cantrell and C. Cosner, *Spatial Ecology via Reaction–Diffusion Equations* (John Wiley & Sons, 2004).
20. J. Keener and J. Sneyd, *Mathematical Physiology: II: Systems Physiology* (Springer, 2009).
21. J. Xin, *SIAM Rev.* **42**, 161 (2000).
22. S.-N. Chow, Lattice dynamical systems, in *Dynamical Systems* (Springer, 2003), pp. 1–102.
23. C. M. Elliott and Z. Songmu, *Arch. Ration. Mech. Anal.* **96**, 339 (1986).
24. P. P. de Oliveira, *J. Cell. Autom.* **9**, 357 (2014).
25. B. Bollobás and B. Bollobas, *Modern Graph Theory* (Springer Science & Business Media, 1998).
26. R. Albert and A.-L. Barabási, *Rev. Mod. Phys.* **74**, 47 (2002).