An extremal eigenvalue problem for a two-phase conductor in ball

Shape derivative for a two-phase eigenvalue problem and optimal configurations in a ball.

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   - The problem
   - Background Research

2 Existence
   - Direct Method
   - $N$ dimensions: total symmetry case

3 Characterization
   - Conjecture
   - $\chi'_1$: Derivation with respect to the domain
   - Numerical Experiments

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4. Conclusions
Introduction

- Let $\Omega$ be a bounded domain, $0 < \alpha < \beta$.
- $\omega \subset \Omega$ region where is placed the material $\beta$.

Spectrum problem

If $\nu := \alpha \chi_{\Omega \setminus \omega} + \beta \chi_{\omega}$, the spectral conductivity problem

$$
\begin{cases}
-\text{div} (\nu \nabla u) = \lambda u & \text{in } \Omega \\
u u = 0 & \text{on } \partial \Omega
\end{cases}
$$

has the first eigenvalue

$$
\lambda^1(\nu) = \inf_{u \in H^1_0(\Omega), u \neq 0} \frac{\int_{\Omega} \nu |\nabla u|^2}{\int_{\Omega} u^2}.
$$

(1)
The problem

Optimization Problem

- We fix the **quantity of material**, $|\omega| = m$.
- Admissible set:
  \[
  \mathcal{A} = \{ \nu \mid \nu = \alpha \chi_{\Omega \setminus \omega} + \beta \chi_{\omega}, \omega \text{ measurable, } |\omega| = m \}.
  \] (3)
- We are interested in studying
  \[
  \inf_{\nu \in \mathcal{A}} \lambda^1(\nu).
  \] (4)
Interesting Questions

- Is it possible to find a classical minimizer, i.e., which belongs to $A$?
- If it is possible, give characterizations.
The problem

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- If it is possible, give characterizations.
Background Research

- **Krein (1955) [8]:** Unidimensional case. It exists a classical solution, completely characterized.

- **Alvino-Lions-Trombetti (1989) [2]:** In the $N$ dimensional case, they proved the existence of a classical radially symmetrical solution in a ball.

- **Murat-Tartar (1970-80) [9]:** In general, the solutions of these kind of problems are coefficients with micro-structure or homogenized ones: The optimal materials have to be finely merged.

- **Cox-Lipton (1996) [5]:** They analyzed the relaxed problem for an optimal solution with micro-structure.
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4. **Conclusions**
Direct Method of the Calculus of Variations

If we could find a topology in $\mathcal{A}$ such that:

- $\{\nu \mid \lambda^1(\nu) \leq c\}$ were relatively compacts,
- and $\lambda^1(\nu)$ were l.s.c
- Then $\lambda^1(\nu)$ would reach the minimum in $\text{cl}(\mathcal{A})$
Direct Method

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Direct Method

**Weak-* Topology**

- The feasible set $\mathcal{A}$ is relatively compact for the weak-* topology in $L^\infty(\Omega)$.
- But $\lambda^1(\nu)$ is not l.s.c for this topology.
- If it were, we would have
  \[ \int_{\Omega} \nu_n \nabla u_n \nabla v \rightarrow \int_{\Omega} \nu \nabla u \nabla v, \]
  For some $\nu_n \rightharpoonup \nu$ and $u_n \rightharpoonup u$, where $u_n$ are the eigen-functions associated to $\nu_n$. This statement, a priori, is not true.
It is not easy to find classical solutions in the general case. (Murat-Tartar, Homogenization theory)

If we had **complete symmetry**: → we could reduce it to a **one-dimensional** problem.

Alvino et. al. solved it. We gave a new proof: Could it work in other kind of domain symmetries.

In the proof we used symmetrization tools: **Schwarz rearrangements**.
$N$ dimensions: total symmetry case

**Schwarz rearrangement**

- If $D \subseteq \Omega$, the *Schwarz rearrangement* $D^*$ is the ball centered at the origin such that $|D| = |D^*|$.
- For $f : \Omega \rightarrow \mathbb{R}$, its rearrangements is given by
  
  $$f^*(x) = \sup \{ c \mid x \in \{ f \geq c \}^* \}.$$

- $f^*$ is radially symmetrical.

**Equi-measurability:**

$$|\{ f \geq c \}| = |\{ f^* \geq c \}|.$$

$$\Rightarrow \int_{\Omega^*} (u^*)^2 = \int_{\Omega} u^2$$
How to diminish the eigenvalue?

\[
\frac{\int_{\Omega} \nu |\nabla u|^2}{\int_{\Omega} u^2} \geq \frac{\int_{\Omega} \tilde{\nu} |\nabla u^*|^2}{\int_{\Omega} (u^*)^2}.
\]

(5)

- \(\tilde{\nu}\) some “rearrangement”. Is it feasible?
- \(u^* \in H^1_0(\Omega)\)? ✓

If (5) is true and \(\tilde{\nu}\) feasible:

→ We reduce the problem to a one-dimensional.

Besides, we still have the topology problem.
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Reformulation of $\mathcal{A}$

Feasible set $\mathcal{A} = \{\nu \mid (\nu^{-1})^* = (\theta^{-1})^*\}$.

- $\mathcal{A}^{-1} = \{\nu^{-1} \mid \nu \in \mathcal{A}\}$
- $\cl(\mathcal{A}^{-1})$ is convex and weak-\* compact.
- Extreme points of $\cl(\mathcal{A}^-) = \mathcal{A}^{-1}$ [2].
And in fact, we have an inequality like (5):

**Theorem (Conca-Mahadevan-Sanz [3], [2])**

Let $\nu \in \mathcal{A}$ and $u \in H^1_0(\Omega)$. There exists $\tilde{\nu}^{-1} \in cl(\mathcal{A}^{-1})$ radially symmetrical such that

$$
\int_{\Omega} \tilde{\nu} |\nabla u^*|^2 \leq \int_{\Omega} \nu |\nabla u|^2,
$$

(6)

where $u^*$ is the Schwarz rearrangement of $u$. 
Topological properties:

- $\lambda^1(\nu)$ is continuous for $\nu$ radially symmetrical, with the weak-* convergence of $\nu^{-1}$.
- $\mathcal{A}$ is relatively compact for this topology.
- The minimizer $\tilde{\nu}$ is such that $\tilde{\nu}^{-1} \in cl(\mathcal{A}^{-1})$.

Classical solution in a Ball:

\[
\frac{1}{\lambda(\nu)} = J(\nu^{-1}) \quad \text{with } J(\cdot) \text{ convex.} \tag{7}
\]

- $\min_{\nu^{-1} \in cl(\mathcal{A}^{-1})} \lambda(\nu) \leftrightarrow \max_{\nu^{-1} \in cl(\mathcal{A}^{-1})} J(\nu^{-1})$
- The solution $\nu^{-1}$ is an extreme point: It lives in $\mathcal{A}^{-1}$.

The solution $\nu$ of our problem lives in $\mathcal{A}$. 
### Topology: weak-* convergence of $\nu^{-1}$

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Conjecture

- Domain: Ball in $\mathbb{R}^N$
- $\lambda^1(\nu)$ is minimized in a classical radially symmetric configuration.

Conjecture

Optimal Solution: Distribute the material $\beta$ in the center.
We proceed with the tools of derivation with respect to the domain: They measure the variations in values that change when the domain is being perturbed.

The set of admissible domains doesn’t have an vectorial space structure.

If \( \omega \) is a region,

\[
\theta : \mathbb{R}^N \rightarrow \mathbb{R}^N,
\]

\[
\omega_t = (Id + t\theta)(\omega),
\]

If \( u = u(\omega) \Rightarrow u_t := u(\omega_t) \)
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If \( u = u(\omega) \) \( \Rightarrow u_t := u(\omega_t) \)
Formulas

- **Total Derivative:**
  \[ \dot{u} = \lim_{t \to 0} \frac{u_t \circ (Id + t\theta) - u}{t}. \quad (8) \]

- **Local Derivative:**
  \[ u' = \lim_{t \to 0} \frac{u_t - u}{t}. \quad (9) \]

- **Relation:**
  \[ u' = \dot{u} - \theta \cdot \nabla u. \quad (10) \]
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Perturbed problem

- $\nu_0, \omega_0$: initial material distribution.
- $(\lambda_0^1, u_0)$: eigen-pair of the non perturbed problem.
- $\nu_t = \alpha \chi_{\Omega \setminus \omega_t} + \beta \chi_{\omega_t}$: material distribution.

Spectral perturbed problem:

\[
\begin{aligned}
-\text{div} (\nu_t \nabla u_t) &= \lambda_t^1 u_t &\text{in} &\Omega \\
 u_t &= 0 &\text{on} &\partial\Omega
\end{aligned}
\]  

$(\lambda_t^1, u_t)$: first eigen-pair of the perturbed problem.
Perturbed problem

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\quad u_t = 0 & \text{on} \quad \partial \Omega
\end{cases}
$$

$(\lambda^1_t, u_t)$: first eigen-pair of the perturbed problem.
Existence and Formula for $\lambda'_1(\omega_0; \theta)$

Theorem (Conca, Mahadevan, Sanz [4])

- $\lambda'_1(\omega_0; \theta)$ exists.

$$\lambda'_1(\omega_0; \theta) = \int_{\partial \omega_0} \left[ \nu_0 |\nabla u_0|^2 \right] \theta \cdot ndS + 2 \int_{\partial \omega_0} \left[ \theta \cdot \nabla u_0 \right] \nu_0 \nabla u_0 \cdot ndS,$$

where $[f]$ denotes the jump of $f$.

In the proof of existence: Implicit Function Theorem.
Existence and Formula for $\lambda'_1(\omega_0; \theta)$

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\]

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Existence and Formula for $\lambda_1'(\omega_0; \theta)$

**Theorem (Conca, Mahadevan, Sanz [4])**

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\]

where $[f]$ denotes the jump of $f$.

In the proof of existence: **Implicit Function Theorem.**
Existence Scheme $\lambda'_1(\omega_0; \theta)$

We have seen:

\[
\begin{cases}
-\text{div}(\nu(\omega_t)\nabla u_t) = \lambda^1_t u_t & \text{in } \partial\Omega \\
u_t = 0 & \text{on } \partial\partial\Omega \\
y \omega_t = (Id + t\theta)(\omega) = \Phi_t(\omega)
\end{cases}
\]

smooth and invertible for $t$ small enough. Change of variables:

\[
\begin{cases}
-\text{div}(\nu(\omega_t) \circ \Phi_t) A_t \nabla (u_t \circ \Phi_t) = \lambda^1_t (u_t \circ \Phi_t) J(\Phi_t) & \text{in } \Omega \\
u_t \circ \Phi_t = 0 & \text{on } \partial\Omega
\end{cases}
\]

where $A_t = D\Phi_t^{-1}(D\Phi_t^{-1})^T J(\Phi_t)$

We have $\nu(\omega_t) \circ \Phi_t = \nu_0 \quad \forall \ t$. The normalization constraint can be written in the form

\[
\int_{\Omega} |u_t \circ \Phi_t|^2 J(\Phi_t) dx = 1
\]
$\lambda'_t$: Derivation with respect to the domain

- $(\lambda^1_t, u_t)$ is a normalized eigen-pair $\Leftrightarrow (\lambda^1_t, u_t \circ \Phi_t)$ satisfies the former equations.

- If $(\lambda^1_t, u_t \circ \Phi_t)$ is a smooth zeros curve of the function $F$ around $(0, \lambda^1_0, u_0)$, where

$$F(t, \lambda, v) = \left( -\text{div} (\nu A_t \nabla v) - \lambda v, \int_{\Omega} |v|^2 J(\Phi_t) dx - 1 \right)$$

$\Rightarrow (\lambda'_t(\omega; \theta), \dot{u})$ exists in $\mathbb{R} \times H^1(\Omega)$

- IFT $\Rightarrow$ The smooth curve exists.

We verify the IFT hypothesis:

- $F : \mathbb{R} \times \mathbb{R} \times H^1_0(\Omega) \longrightarrow \mathbb{R}$ is differentiable.

- $F_{\lambda,v}(0, \lambda^1(\omega), u_0) : \mathbb{R} \times H^1_0(\Omega) \longrightarrow H^{-1} \times \mathbb{R}$ in invertible with continuous inverse.
\( \lambda'_1 \): Derivation with respect to the domain

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$\lambda_1'$: Derivation with respect to the domain

$\lambda_1'(\omega; \theta)$ Contribution: Its sign

- To indicate descent directions of $\lambda^1$.
- To reinforce the conjecture in a numerical way.
- To provide future descent algorithms for other type of geometries.
Numerical Experiments

Experiments in the plane $\mathbb{R}^2$

- Concentric Rings
- Displaced Discs
- Concentric Squares
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Matlab and Freefem

**Anillos Concéntricos**
- First EigenValue Vs Internal Radius. (Concentric Rings)
- \( a = 1 \)
- \( b = 200 \)
- Proportion of \( b = 0.5 \)
- Number of Samples = 50

**Discos Desplazados**
- First EigenValue Vs Displacement of the Center
- \( a = 1 \)
- \( b = 200 \)
- Proportion of \( b = 0.9 \)
- Number of Samples = 100

**Cuadrados Concéntricos**
- First EigenValue Vs Internal Radius. (Concentric Squares)
- \( a = 1 \)
- \( b = 200 \)
- Proportion of \( b = 0.5 \)
- Number of Samples = 100
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4 Conclusions
Conclusions and projections

- We gave a new proof for a classical radially symmetrical solution.
- We expect to understand the problem in domains with less symmetries, such as the case of squares and stars.
- The conjecture: It is optimal to distribute $\beta$ in the center of the ball.
- We gave arguments and evidence:
  - Derivative with respect to the domain of the first eigenvalue.
  - Numerical experiments in the plane.
- We expect to provide a deeper view in the analysis of the $\lambda'_1(\omega; \theta)$
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Thanks!